# LECTURE 5 - 1/17/17

EXAMPLES OF LINEAR SYSTEMS

①: ECONOMIC MODELS.

IDEA: DIVIDE ECONOMY INTO "SECTORS", LOOK AT HOW MUCH EACH SECTOR BUYS FROM EACH OTHER ONE. ASSUME SYSTEM IS <u>BALANCED</u>: MONEY IN = MONEY OUT.

EXAMPLE FROM BOOK: 3 SECTORS

COAL, ELECTRIC, STEEL.

<u>EXCHANGE MODEL</u>: HAVE OUTPUT TABLE

PRODUCTION

| SELLS TO | COAL | ELECTRIC | STEEL |
|---|---|---|---|
| COAL | .0 | .4 | .6 |
| ELECTRIC | .6 | .1 | .2 |
| STEEL | .4 | .5 | .2 |

.4 IN TOP CENTER: .4 OF PRODUCTION OF ELECTRICITY (40%) IS SOLD TO COAL INDUSTRY

LET $P_C$ = TOTAL MONETARY VALUE OF COAL PRODUCED.

$P_E$ = " ELECTRICITY

$P_S$ = " STEEL

COAL INDUSTRY **MAKES** $p_C$, **SPENDS** $.4p_E + .6p_S$.
ELECTRIC **MAKES** $p_E$ **SPENDS** $.6p_C + .1p_E + .2p_S$
STEEL **MAKES** $p_S$ **SPENDS** $.4p_C + .5p_E + .2p_S$.

**ASSUMPTION**: MAKES = SPENDS.

GET SYSTEM OF LINEAR EQUATIONS:

$$p_C = .4p_E + .6p_S$$
$$p_E = .6p_C + .1p_E + .2p_S$$
$$p_S = .4p_C + .5p_E + .2p_S.$$

REWRITE:

$$p_C - .4p_E - .6p_S = 0$$
$$-.6p_C + .9p_E - .2p_S = 0$$
$$-.4p_C - .5p_E + .8p_S = 0$$

SOLVE: FIND $p_S$ IS <u>FREE VARIABLE</u>

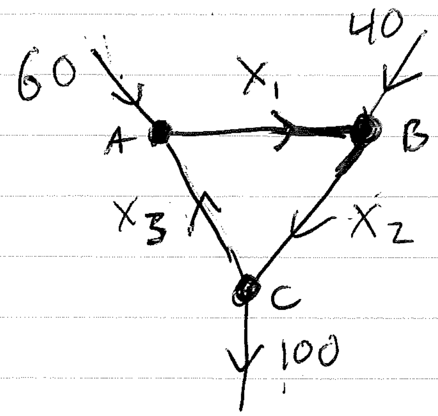
$$\begin{bmatrix} p_C \\ p_E \\ p_S \end{bmatrix} \approx \begin{bmatrix} .94p_S \\ .85p_S \\ p_S \end{bmatrix}$$

APPROXIMATE; EXACT ANSWERS ARE A MESS.

SO: ONE POSSIBLE SOLUTION: $p_S = 100$ (\$100 MILLION)

$$\Rightarrow p_E \approx 85$$
$$p_C \approx 94$$

## NETWORK FLOWS




DIRECTED

SYSTEM OF NODES AND EDGES

(INTERSECTIONS) (ROADS)

ONE-WAY

AT EACH NODE, INPUT = OUTPUT.

NODE
A: $60 + x_3 = x_1$
B: $40 + x_1 = x_2$
C: $x_2 = 100 + x_3$

REARRANGE:
$$x_1 - x_3 = 60$$
$$x_1 - x_2 = -40$$
$$x_2 - x_3 = 100$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -1 & 60 \\ 1 & -1 & 0 & -40 \\ 0 & 1 & -1 & 100 \end{array}\right] \rightsquigarrow \boxed{\begin{array}{l} x_1 = x_3 + 60 \\ x_2 = x_3 + 100 \\ x_3 \ \text{FREE.} \end{array}} \begin{array}{l} \geq 0 \\ \geq 0 \\ \geq 0 \end{array}$$

ARE ALL POSSIBLE VALUES OF $x_3$ SENSIBLE?
NO! ALL VALUES SHOULD BE $\geq 0$
(NO NEGATIVE FLOW – NO GOING BACKWARDS DOWN 1-LANE ROADS).

ONLY $\boxed{x_3 \geq 0}$ MAKES SENSE.

POSSIBLE SOLUTIONS: $x_3 = 0$ $\begin{pmatrix} x_1 = 60 \\ x_2 = 100 \end{pmatrix}$.

$x_3 = 10$ $\begin{pmatrix} x_1 = 70 \\ x_2 = 110 \end{pmatrix}$

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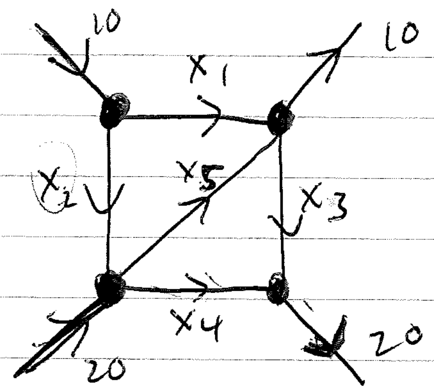
EXAMPLE:




EQUATIONS FOR EACH INTERSECTION:

$$10 = x_1 + x_2$$
$$20 + x_2 = x_4 + x_5$$
$$x_1 + x_5 = x_3 + 10$$
$$x_3 + x_4 = 20$$

SOLVE, GET 2 FREE VARIABLES. ($x_4$ & $x_5$)

$\rightsquigarrow$
$$\begin{aligned} x_1 &= 30 - x_4 - x_5 && \geq 0 \\ x_2 &= x_4 + x_5 - 20 && \geq 0 \\ x_3 &= 20 - x_4 && \geq 0 \\ \boxed{x_4} & && \geq 0 \\ \boxed{x_5} & && \geq 0 \end{aligned}$$

$$\boxed{\begin{aligned} 0 \leq x_4 \leq 20 \\ 0 \leq x_5 \leq 30 \end{aligned}}$$

$$20 - x_4 \geq 0$$
$$20 \geq x_4$$

(REALLY: $0 \leq x_5 \leq 30 - x_4$).

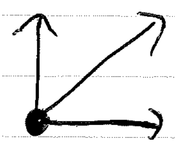
$\boxed{x_2 \leq 10}$

$$20 + x_2 = x_4 + x_5$$
$$x_5 = 20 + x_2 - x_4 \leq 20 + 10 - x_4$$

LINEAR INDEPENDENCE (REALLY IMPORTANT CONCEPT FOR REST OF SEMESTER)

BASIC IDEA: WHEN IS A LIST OF VECTORS "REDUNDANT"?

EXAMPLE: $\mathbb{R}^2 = \text{SPAN}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right)$.



LIST IS REDUNDANT: THIRD VECTOR COMES FROM OTHER TWO:

$$\begin{bmatrix}1\\1\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}.$$

ANY LINEAR COMBINATION OF $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ IS JUST A LINEAR COMBINATION OF $\begin{bmatrix}1\\0\end{bmatrix} \begin{bmatrix}0\\1\end{bmatrix}$

$$x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} + z\begin{bmatrix}1\\1\end{bmatrix}$$
$$= x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} + z\left(\begin{bmatrix}1\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right)$$
$$= (x+z)\begin{bmatrix}1\\0\end{bmatrix} + (y+z)\begin{bmatrix}0\\1\end{bmatrix}.$$

SO: $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ BEING "REDUNDANT" (LINEARLY DEPENDENT)

MEANS THERE'S A VECTOR YOU CAN THROW OUT AND KEEP THE SAME SPAN.

IN THIS EXAMPLE YOU CAN THROW OUT ANY OF THEM:

$$\text{SPAN}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = \mathbb{R}^2$$

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$$\text{SPAN}\left(\begin{bmatrix}0\\1\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = \mathbb{R}^2.$$

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B. $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}$ IS REDUNDANT BUT CAN'T THROW OUT $\begin{bmatrix}1\\1\end{bmatrix}$.

$$\text{SPAN}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = \mathbb{R}^2$$

$$\text{SPAN}\left(\begin{bmatrix}2\\0\end{bmatrix}, \begin{bmatrix}1\\1\end{bmatrix}\right) = \mathbb{R}^2$$

$$\text{SPAN}\left(\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}\right) \neq \mathbb{R}^2$$