LECTURE 4 – 1/12/17

- QUIZZES WILL BE RETURNED THROUGH MATH LEARNING CENTER (LSK 301/302).
  (YOU MUST PICK UP YOUR QUIZ TO SEE YOUR GRADE)

- PRACTICE PROBLEMS FOR QUIZ 2 ONLINE
- EXTRA PRACTICE PROBLEMS ON WEBWORK.
- REMINDER: THESE NOTES SCANNED & PUT ONLINE.

LEFT OFF ON TUESDAY WITH MULTIPLYING
A MATRIX BY A VECTOR

$m \times \underline{n}$ $\qquad$ $\underline{n} \times 1$

\# COLUMNS OF MATRIX = \# ROWS OF VECTOR

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 + 3x_2 \\ 2x_1 \\ -x_1 + 2x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix}$$

"ROW-COLUMN RULE" $\qquad$ LINEAR COMBINATION OF COLUMNS.

3 EQUIVALENT WAYS TO WRITE A SYSTEM OF EQUATIONS:

$$\begin{cases} x_1 + 3x_2 = 1 \\ 2x_1 = 2 \\ -x_1 + 2x_2 = 3 \end{cases}$$

SYSTEM OF LINEAR EQUATIONS

$\Updownarrow$

$$x_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$(x_1 a_1 + x_2 a_2 = b)$$

VECTOR EQUATION.

$\Updownarrow$

$$\begin{bmatrix} 1 & 3 \\ 2 & 0 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$\underline{Ax = b}$$

MATRIX PRODUCT EQUATION.

HERE: b IS ALREADY FIXED $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$

WHAT IF WE WANT TO LOOK AT A LOT OF b's ?

NICEST SITUATION: GIVEN A MATRIX A, $(m \times n)$
THE SYSTEM $Ax = b$ IS ALWAYS CONSISTENT
(FOR ALL $\underline{b \text{ IN } \mathbb{R}^m}$)

<u>THEOREM</u>: FOR A GIVEN $m \times n$ MATRIX A,
THE FOLLOWING ARE EQUIVALENT:

1. FOR EVERY b IN $\mathbb{R}^m$, $Ax = b$ IS CONSISTENT.
2. EVERY b IN $\mathbb{R}^m$ IS A LINEAR COMBINATION OF THE COLUMNS $a_1, \ldots, a_n$ OF A.
3. THE SPAN OF THE COLUMNS $\text{SPAN}(a_1, \ldots, a_n)$ IS ALL OF $\mathbb{R}^m$
4. THE RREF OF THE $\boxed{\text{COEFFICIENT MATRIX}}$ A HAS A PIVOT IN EVERY <u>ROW</u>.

(ALL OF 1, 2, 3, 4 ARE TRUE, OR ALL ARE FALSE).

<u>EXAMPLES</u>:

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$Ax = b \quad = \quad \begin{bmatrix} 1 \\ 2 \end{bmatrix} [x_1] = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \rightsquigarrow \begin{matrix} x_1 = b_1 \\ 2x_1 = b_2. \end{matrix}$$

x_1 VARIABLE b_1, b_2 CONSTANTS.

1 IS FALSE: $Ax = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ IS NOT SOLVABLE.

2 IS FALSE: NOT EVERYTHING IN $\mathbb{R}^2$ IS A LINEAR COMBINATION OF $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$

CONTINUED EXAMPLE

③ IS FALSE: $\text{SPAN}\left(\begin{bmatrix}1\\2\end{bmatrix}\right) \neq \mathbb{R}^2$

④ IS FALSE: RREF OF $\begin{bmatrix}1\\2\end{bmatrix} = \begin{bmatrix}1\\0\end{bmatrix}$ DOES NOT HAVE A PIVOT IN ROW 2.

WHY IS THE THEOREM TRUE?

① ⟺ ② THE EQUATION $Ax=b$ IS $x_1a_1 + \cdots + x_na_n = b$.
SO ALWAYS BEING SOLVABLE
= ALL b BEING LINEAR COMBINATIONS.

② ⟺ ③ SPAN IS SET OF EVERYTHING WE CAN GET AS A LINEAR COMBINATION.

IF ④ IS TRUE, THEN LOOK AT $Ax=b$ FOR ANY b.
AUGMENTED MATRIX $[A \mid b]$

ROW REDUCING $[A \mid b]$
GIVES SOMETHING WITH PIVOT IN EVERY ROW, TO THE LEFT OF THE VERTICAL LINE

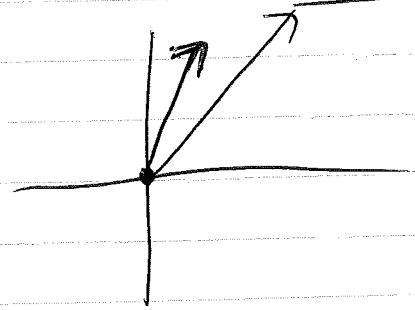
SO NO ROOM FOR PIVOT IN LAST COLUMN, SO CONSISTENT!

EXAMPLE: ALL TRUE: $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

(4) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} \underline{1} & 0 \\ 0 & \underline{1} \end{bmatrix}$

TRUE

(3) $\text{SPAN}\left(\begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \end{bmatrix} \right) = \mathbb{R}^2$ TRUE



(2) EVERY VECTOR IN $\mathbb{R}^2$ IS A LINEAR COMBINATION OF $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$ $\begin{bmatrix} 2 \\ 4 \end{bmatrix}$

(1) $Ax = b$ ALWAYS SOLVABLE:

$$\left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 3 & 4 & b_2 \end{array}\right] \longrightarrow \left[\begin{array}{cc|c} 1 & 2 & b_1 \\ 0 & -2 & b_2 - 3b_1 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 0 & b_2 - 2b_1 \\ 0 & 2 & b_2 - 3b_1 \end{array}\right] = \left[\begin{array}{cc|c} \underline{1} & 0 & b_2 - 2b_1 \\ 0 & \underline{1} & \frac{1}{2}(b_2 - 3b_1) \end{array}\right]$$

CONSEQUENCE: IF THIS IS TRUE, NEED $n = \#$ COLUMNS $\geq m = \#$ ROWS.

(LOOK AT (4)).

WHAT IF A DOES <u>NOT</u> HAVE PIVOTS IN EVERY ROW? (④ IS FALSE). SHOULD MEAN $Ax=b$ IS <u>NOT</u> ALWAYS CONSISTENT (① IS FALSE).

WHAT DOES IT MEAN FOR A ROW TO NOT HAVE A PIVOT? A ROW OF ALL 0'S.

EX: $R = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

CLAIM $Rx=b$ IS <u>NOT</u> ALWAYS CONSISTENT

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

CAN'T HAVE A SOLUTION.

IF A IS NOT ALREADY IN RREF:

$A = \begin{bmatrix} 1 & 2 & 8 \\ -2 & 1 & -1 \\ -1 & -1 & -5 \end{bmatrix}$ ROW REDUCES TO R ABOVE.

CLAIM $Ax=b$ NOT ALWAYS SOLVABLE – NEED TO FIND SINGLE b THAT FAILS.

$$\left[\begin{array}{ccc|c} 1 & 2 & 8 & b_1 \\ 2 & 1 & -1 & b_2 \\ -1 & -1 & -5 & b_3 \end{array}\right] \xrightarrow{\text{ROW REDUCTION}} \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

REVERSE STEPS TO FIND b.

SOLUTION SETS OF LINEAR SYSTEMS

GIVEN A SYSTEM OF LINEAR EQUATIONS $Ax = b$, WHAT DOES THE SOLUTION SET LOOK LIKE?

SOLUTION SET: $\{ \mathbf{v}$ IN $\mathbb{R}^n$: $A\mathbf{v} = b$, I.E $\mathbf{v}$ IS A SOLUTION $\}$.

SIMPLE EXAMPLE: $A = \begin{bmatrix} 2 & -1 \end{bmatrix}$ $b = \begin{bmatrix} 0 \end{bmatrix}$.

SOLUTION SET : SUBSET OF $\mathbb{R}^2$ OF ALL VECTORS $\begin{bmatrix} x \\ y \end{bmatrix}$ WITH

$\begin{bmatrix} 2 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}$, I.E. $\underline{2x - y = 0}$.

THIS IS ALL VECTORS OF THE FORM $\begin{bmatrix} x \\ 2x \end{bmatrix}$,

I.E SPAN$\left(\begin{bmatrix} 1 \\ 2 \end{bmatrix} \right)$; A LINE THROUGH THE ORIGIN.

IN GENERAL: ANY SYSTEM $Ax = 0$ IS CALLED A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS.

NICE THING: HOMOGENEOUS SYSTEMS ARE ALWAYS CONSISTENT.

ALWAYS HAVE SOLUTION $x = 0$.
(THE TRIVIAL SOLUTION)

THEN: SOLUTION SET COULD BE JUST $\{0\}$
(ONLY THE TRIVIAL SOLUTION
NO FREE VARIABLES)

OR COULD HAVE NONTRIVIAL SOLUTIONS
(HAVE FREE VARIABLES).

IMPLICIT

ANOTHER EXAMPLE: EQUATION $\boxed{x_1 - 2x_2 - 3x_3 = 0}$

SOLVE: FIND x_2, x_3 FREE, $x_1 = 2x_2 + 3x_3$,
SO SOLUTIONS ARE

EXPLICIT; PARAMETRIC VECTOR FORM

$$\begin{bmatrix} 2x_2 + 3x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$$

SO SOLUTION SET IS $\text{SPAN}\left(\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \right)$ ← A PLANE IN $\mathbb{R}^3$

GENERAL FACT: THE SOLUTION SET OF A HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS CAN BE WRITTEN AS A SPAN,

SPAN$(v_1, \dots, v_k)$, k = NUMBER OF FREE VARIABLES.

NON HOMOGENEOUS SYSTEMS: $Ax = b$, $b \neq 0$.

$$\boxed{x_1 - 2x_2 - 3x_3 = 1}$$

SOLVE, FIND SOLUTIONS ARE.

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} \quad \text{FOR ALL } x_2, x_3.$$

PARAMETRIC VECTOR FORM

NOT A SPAN: (PLANE NOT THROUGH THE ORIGIN).

WHAT IS THIS SET? IT'S A VECTOR "PLUS" A SPAN

= A SPAN SHIFTED (TRANSLATED) BY A VECTOR.

SUMMARY:

- A HOMOGENEOUS SYSTEM $Ax=0$ ALWAYS HAS SOLUTIONS. SOLUTION SET IS A SPAN.
- A NONHOMOGENEOUS SYSTEM $Ax=b$ MAY OR MAY NOT HAVE SOLUTIONS.

  - IF NO SOLUTIONS: SOLUTION SET EMPTY.
  - IF THERE ARE SOLUTIONS, SOLUTION SET IS A TRANSLATE OF THE SOLUTION SET FOR $Ax=0$. (TRANSLATE OF A SPAN).

REPHRASING: IF v_0 IS ONE SOLUTION TO $Ax=b$, THEN ALL SOLUTIONS ARE OF THE FORM v_0+v_h, v_h A SOLUTION OF $Ax=0$ (h FOR "HOMOGENEOUS").

SO: TO UNDERSTAND NONHOMOGENEOUS SYSTEMS $Ax=b$, IMPORTANT TO UNDERSTAND CORRESPONDING HOMOGENEOUS SYSTEM $Ax=0$.

IN BOTH CASES, GET PARAMETRIC VECTOR FORM FOR SOLUTION SETS (IF NONEMPTY).

$$v_0 + x_1 v_1 + \cdots + x_k v_k, \qquad x_1, \ldots x_k$$

FREE VARIABLES