LECTURE 24 - 4/6/17

ADMINSTRATIVE THINGS:

- FINAL IS APRIL 25 (TUESDAY)
  12:00, "SRC A-B-C"
- INFO ON WEBSITE SHORTLY.
- I'LL RETRIEVE QUIZZES FROM MLC TOMORROW.
  (GET BACK DURING OFFICE HOURS)
- OFFICE HOUR SCHEDULE WILL BE ON WEBSITE.

LEAST SQUARES:

SAY WE HAVE AN (INCONSISTENT) LINEAR SYSTEM $Ax=b$. WANT AN APPROXIMATE SOLUTION.

A <u>LEAST SQUARES SOLUTION</u> IS AN x THAT MINIMIZES

$$\|Ax-b\|$$

GEOMETRICALLY: TAKE ORTHOGONAL PROJECTION $\hat{b}$ OF b TO COL(A), SOLVE $\underline{Ax=\hat{b}}$

SOMETIMES CAN FIND LEAST SQUARES SOLUTION FROM THIS.

EX: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \\ 0 & -2 \end{bmatrix}$ $\quad b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\begin{array}{cc} \uparrow & \uparrow \\ a_1 & a_2 \end{array}$

$$\underbrace{\begin{aligned} x_1 + x_2 &= 1 \\ -x_1 + x_2 &= 1 \\ -2x_2 &= -1 \end{aligned}}$$

INCONSISTENT.

FIND $\hat{b}$: IN THIS CASE COLUMNS OF A ARE ORTHOGONAL, SO ARE ORTHOGONAL BASIS OF COL(A).

$$\hat{b} = \underbrace{\frac{(a_1 \cdot b)}{(a_1 \cdot a_1)} a_1 + \frac{(a_2 \cdot b)}{(a_2 \cdot a_2)} a_2}_{\text{FORMULA FOR ORTHOGONAL PROJECTIONS.}} = 0 \cdot a_1 + \frac{4}{6} a_2$$

$$= \boxed{\begin{bmatrix} 2/3 \\ 2/3 \\ -4/3 \end{bmatrix} = \hat{b}}$$

SOLUTION TO $Ax = \hat{b}$ ALREADY HERE!

$\boxed{x = \begin{bmatrix} 0 \\ 2/3 \end{bmatrix}}$ IS THE LEAST SQUARES SOLUTION

THAT WORKS <u>IF</u> COLUMNS OF A ARE ORTHOGONAL.
MUCH MORE WORK IF THEY AREN'T!

ALGEBRAIC TRICK THAT GIVES US A BETTER METHOD IN GENERAL:

<u>THEOREM</u>: THE LEAST SQUARES SOLUTIONS TO $Ax=b$ ARE THE ACTUAL SOLUTIONS TO

$$(A^TA)x = (A^Tb)$$

<u>Ex</u>:

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix}$$

$$A^TA = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 8 & 10 \\ 8 & 20 & 26 \\ 10 & 26 & 38 \end{bmatrix}$$

$$A^Tb = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3 & 1 & 1 & 3 \\ 5 & 0 & 2 & 3 \end{bmatrix} \begin{bmatrix} 3 \\ 5 \\ 7 \\ -3 \end{bmatrix} = \begin{bmatrix} 12 \\ 12 \\ 20 \end{bmatrix}$$

SOLVE $(A^TA)x = A^Tb$

$$\left[\begin{array}{ccc|c} 4 & 8 & 10 & 12 \\ 8 & 20 & 26 & 12 \\ 10 & 26 & 38 & 20 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 4 & 5 & 6 \\ 4 & 10 & 13 & 6 \\ 5 & 13 & 19 & 10 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 2 & 4 & 5 & 6 \\ 0 & 2 & 3 & -6 \\ 1 & 5 & 9 & -2 \end{array}\right] \xrightarrow[\text{KEEP GOING}]{} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 10 \\ 0 & 1 & 0 & -6 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

LEAST SQUARES SOLUTION IS

$$x = \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix}$$

PLUG BACK IN: $$Ax = \begin{bmatrix} 1 & 3 & 5 \\ 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 3 & 3 \end{bmatrix} \begin{bmatrix} 10 \\ -6 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 8 \\ -2 \end{bmatrix}$$

SEEMS LIKE A DECENT APPROXIMATION TO b.

TWO QUESTIONS:

1. WHY DOES THIS WORK?

NEED x SUCH THAT Ax = ORTHOG. PROJ OF b. ONTO COL(A)

$\iff$ $Ax-b$ PERPENDICULAR TO COL(A).

USE A FACT THAT $\mathrm{COL}(A)^{\perp} = \mathrm{NUL}(A^T)$

2. WHY IS THIS CONSISTENT ALWAYS?

~~COMES OUT FROM ABOVE~~

FACT: $\mathrm{RANK}(A^TA) = \mathrm{RANK}(A)$.

$\Rightarrow \mathrm{COL}(A^TA) = \mathrm{COL}(A^T)$.

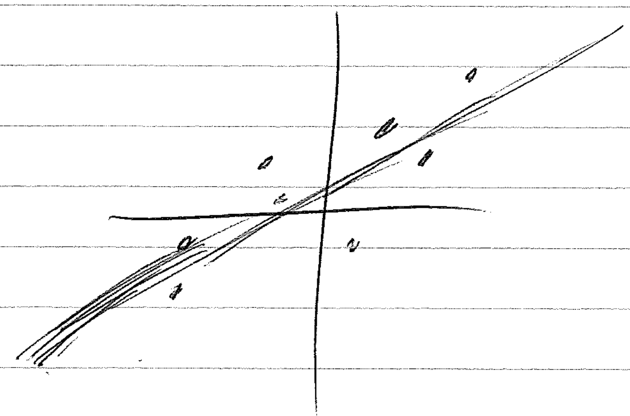
SO A^Tb IS IN $\mathrm{COL}(A^T)$

$\Rightarrow$ IS IN $\mathrm{COL}(A^TA)$

$\Rightarrow$ THERE'S A SOLUTION.

ONE MORE EXAMPLE: LEAST SQUARES AND CURVE FITTING.

IDEA: HAVE DATA POINTS (x_1, y_1)
(x_2, y_2)
$\vdots$
(x_n, y_n)



LIKE TO FIND BEST LINE ~~FOR~~ APPROXIMATING THEM

LINE IS GIVEN BY $y = \alpha x + \beta$

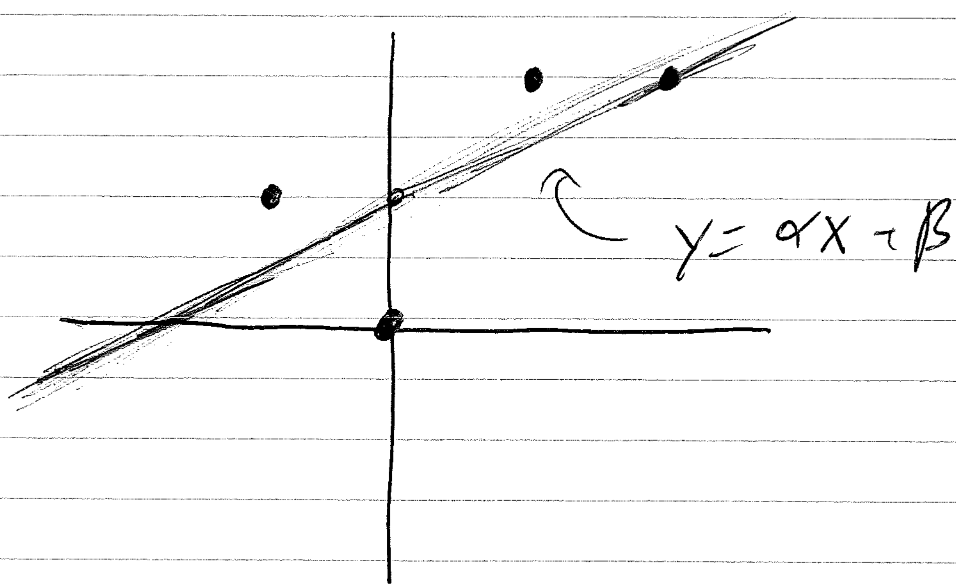
SOLVE FOR α, β.

WELL, HAVE 2 VARIABLES α, β
n EQUATIONS:

$$\begin{aligned} \alpha x_1 + \beta &= y_1 \\ \alpha x_2 + \beta &= y_2 \\ &\vdots \\ \alpha x_n + \beta &= y_n \end{aligned}$$

DO LEAST SQUARES!

EXAMPLE: POINTS $(-1,1)$, $(0,0)$, $(1,2)$, $(2,2)$



SYSTEM OF EQUATIONS:

$$\begin{aligned}(-1)\alpha + \beta &= 1\\ 0\alpha + \beta &= 0\\ 1\alpha + \beta &= 2\\ 2\alpha + \beta &= 2\end{aligned}$$

$$\underbrace{\begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \alpha \\ \beta \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix}}_{b}$$

$\rightsquigarrow$ LEAST SQUARES SOLUTIONS

$$A^T A = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 0 & 1 \\ 1 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix}$$

$$A^T b = \begin{bmatrix} -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

SOLVE $\begin{bmatrix} 6 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$

GET $\alpha = 1/2 \quad \beta = 1$

BEST LINE: $\boxed{y = \frac{1}{2}x + 1}$

COULD ALSO APPROXIMATE A QUADRATIC EQUATION

$$\alpha x^2 + \beta x + \gamma.$$

MORE ON THIS IN SECTION 5.6 (NOT ON EXAM).

ONE LAST TOPIC : <u>THE SPECTRAL THEOREM</u>

(NOT IN BOOK, NOT ON EXAM)

DEFINITION: A MATRIX IS <u>SYMMETRIC</u> IF $A^T = A$.

EX.
$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix}$$

IMPORTANT PROBLEM IS TO UNDERSTAND THEIR EIGENVECTORS.

FOR SYMMETRIC MATRICES THINGS WORK OUT NICELY.

<u>LEMMA</u>: IF v_1, v_2 ARE EIGENVECTORS OF A SYMMETRIC MATRIX A WITH EIGENVALUES $\lambda_1 \neq \lambda_2$, THEN v_1, v_2 ARE <u>PERPENDICULAR</u>.

WHY? BECAUSE

$$v_1^T A^T v_2 = (A v_1) \cdot v_2 = (\lambda_1 v_1) \cdot v_2 = \lambda_1 (v_1 \cdot v_2)$$
$$\underbrace{v_1^T A v_2}_{\text{EQUAL BECAUSE } A = A^T} = v_1 \cdot (A v_2) = v_1 \cdot (\lambda_2 v_2) = \lambda_2 (v_1 \cdot v_2)$$

$\lambda_1 \neq \lambda_2 \Rightarrow v_1 \cdot v_2 = 0.$

SPECTRAL THEOREM: IF A IS A SYMMETRIC MATRIX, IT IS ORTHOGONALLY DIAGONALIZABLE

1. THERE IS AN ORTHOGONAL BASIS OF $\mathbb{R}^n$ OF EIGENVECTORS OF A. (ALWAYS!)
2. THERE'S AN ORTHOGONAL MATRIX U ($U^{-1} = U^T$) SUCH THAT

$$A = UDU^T$$

OUR CHANGE OF BASIS MATRIX "P".

EX:

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 3 & 1 \\ 2 & 1 & 0 \end{bmatrix} \qquad \det(A - \lambda I) = -(\lambda-4)(\lambda-1)(\lambda+2)$$

EIGENVALUES $\lambda = 4, 1, -2$

EIGENVECTOR $\lambda = 4$

$$\begin{bmatrix} -4 & 1 & 2 \\ 1 & -1 & 1 \\ 2 & 1 & -4 \end{bmatrix} \rightsquigarrow v_1 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

EIGENVECTOR

$\lambda = 1$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix} \qquad v_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$\lambda = -2$

$$\begin{bmatrix} 2 & 1 & 2 \\ 1 & 5 & 1 \\ 2 & 1 & 2 \end{bmatrix} \qquad v_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

BASIS OF EIGENVECTORS:

$$\underbrace{\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{\text{CHECK THEY'RE ORTHOGONAL!}},$$

$$U = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ 2/\sqrt{6} & -1/\sqrt{3} & 0 \\ 1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{bmatrix} \qquad D = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

$$A = U D U^T$$