LECTURE 21 - 3/28/17

INNER PRODUCTS, ORTHOGONALITY, LEAST SQUARES:

INNER PRODUCT = DOT PRODUCT ON $\mathbb{R}^n$

INPUT 2 VECTORS $\longmapsto$ OUTPUT OF A SCALAR.

$$x \cdot y$$

$$\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \cdot \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

CAN ALSO EXPRESS IN TERMS OF MATRIX MULT.

x IS $n \times 1$, x^T IS $1 \times n$

y IS $n \times 1$

$$x^T y = \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} = x \cdot y$$

AS A 1×1 MATRIX = SCALAR.

$$\begin{bmatrix} 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix} = 1 \cdot 1 + 1 \cdot (-1) + 2 \cdot 0 + 3 \cdot 2$$

$$= 1 - 1 + 0 + 6$$

$$= 6$$

~~EX~~

THIS ENCODES A LOT OF EUCLIDEAN GEOMETRY.

E.G. LENGTH.

LENGTH OF $X = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

SHOULD BE $\sqrt{\underbrace{x_1^2 + x_2^2 + x_3^2}_{\text{THIS IS } X \cdot X}}$

SO DEFINE THE LENGTH OF X IN $\mathbb{R}^n$

$$\boxed{\|x\| = \sqrt{X \cdot X}}$$

(NOTE:
$X \cdot X = x_1^2 + x_2^2 + \cdots + x_n^2$
IS ALWAYS ≥ 0,
AND IS ALWAYS > 0
IF $X \neq 0$

LENGTH SATISFIES $\|aX\| = |a| \cdot \|X\|$
a A SCALAR, X A VECTOR.

JUSTIFICATION: $\|aX\| = \sqrt{(aX) \cdot (aX)}$
$= \sqrt{a^2 (X \cdot X)}$
$= \sqrt{a^2} \sqrt{X \cdot X} = |a| \, \|X\|$

IMPORTANT DEFINITION HERE:

A UNIT VECTOR IS ANY WITH LENGTH 1.

ANY VECTOR $(\neq 0)$ CAN BE SCALED TO LENGTH 1

$$v \rightsquigarrow \underbrace{\frac{1}{\|v\|} \cdot v}_{\text{LENGTH } 1}$$

EXAMPLES:

$$v = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \qquad \|v\| = \sqrt{v \cdot v} = \sqrt{9+25} = \sqrt{34}$$

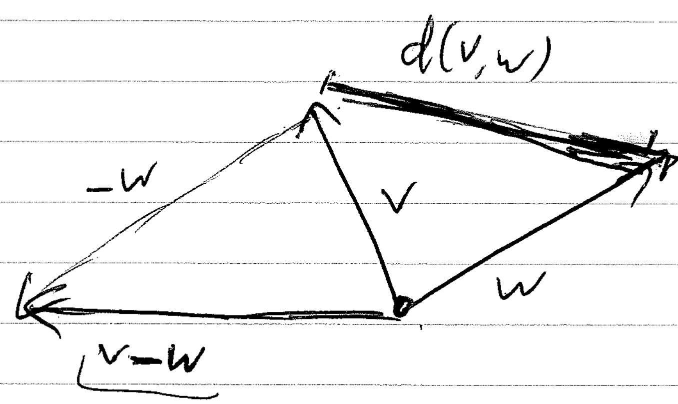
NORMALIZED VERSION IS $\frac{1}{\sqrt{34}} \begin{bmatrix} 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 3/\sqrt{34} \\ 5/\sqrt{34} \end{bmatrix}$

$$v = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \qquad \|v\| = \sqrt{1+4+4} = \sqrt{9} = 3$$

NORMALIZED VERSION IS $\frac{1}{3} \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1/3 \\ 2/3 \\ 2/3 \end{bmatrix}$

LENGTH LETS US DEFINE DISTANCE BETWEEN VECTORS v, w IN $\mathbb{R}^n$

$$\boxed{d(v,w) = \|v-w\|}$$



NOTE: THIS LETS US GENERALIZE GEOMETRY TO $\mathbb{R}^n$

ORTHOGONALITY (= PERPENDICULAR-ITY)

TWO VECTORS x, y IN $\mathbb{R}^n$ ARE ORTHOGONAL IF $\boxed{x \cdot y = 0}$

(THEOREM: THIS AGREES WITH "PERPENDICULAR" IN GEOMETRY)

Ex. 1: $\begin{bmatrix}1\\-1\\2\\-2\end{bmatrix} \cdot \begin{bmatrix}2\\2\\1\\1\end{bmatrix} = 2-2+2-2 = \underline{0}$

PERPENDICULAR IN $\mathbb{R}^4$.

Ex. 2 REMEMBER $A = \begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}$

IS ROTATION BY $90°$ (CCW)

FOR ANY x, x AND Ax SHOULD BE PERPENDICULAR.

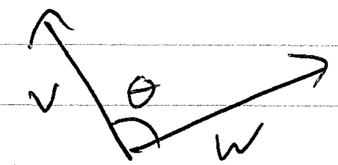
SO LET'S CHECK! $x = \begin{bmatrix}x_1\\x_2\end{bmatrix}$

$$x \cdot Ax = \underbrace{x^T \cdot A \cdot x}_{\text{MATRIX MULT.}} = [x_1 \;\; x_2]\begin{bmatrix}0 & -1\\1 & 0\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$= [x_2 \;\; -x_1]\begin{bmatrix}x_1\\x_2\end{bmatrix}$$

$$= x_1x_2 - x_1x_2 = \underline{0}$$

ANGLES: IN $\mathbb{R}^2$, $\mathbb{R}^3$, IF ANGLE BETWEEN VECTORS v, w IS θ



LAW OF COSINES SAYS

$$v \cdot w = \|v\| \cdot \|w\| \cdot \cos\theta.$$

SO TURN THIS AROUND, ALGEBRAICALLY **DEFINE** THE ANGLE BETWEEN v, w AS

$$\boxed{\theta = \cos^{-1}\left(\frac{v \cdot w}{\|v\| \cdot \|w\|}\right)}$$

INVERSE COSINE = ARCCOSINE

ORTHOGONAL COMPLEMENTS:

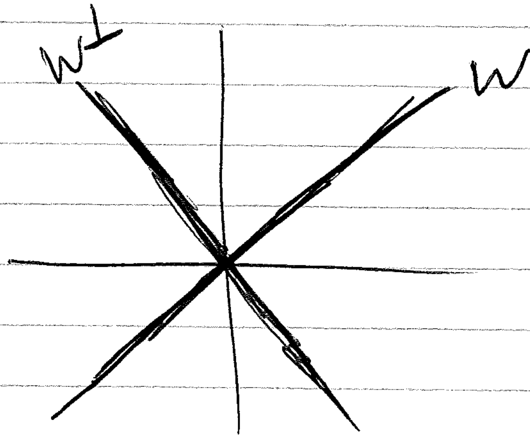
DEFINITION: IF W IS A SUBSPACE IN $\mathbb{R}^n$ SET

$$W^{\perp} = \{\text{ALL } v \text{ IN } \mathbb{R}^n \text{ SATISFYING: } v \cdot w = 0 \text{ FOR ALL } w \text{ IN } W\}$$

ORTHOGONAL COMPLEMENT

"W PERP"

I.E. v THAT ARE PERPENDICULAR TO ALL OF W.



FIRST FACT: $W^\perp$ IS A SUBSPACE ITSELF. ✓

CHECK 3 THINGS:

- 0 IS IN $W^\perp$ (SURE, $0 \cdot v = 0$ ALWAYS)
- SCALAR MULT: IF a IS A SCALAR AND v IS IN $W^\perp$, av IS IN $W^\perp$
  (COMES FROM $(av) \cdot w = a(v \cdot w)$)
- ADDITION: IF v_1, v_2 ARE IN $W^\perp$ SO IS $v_1 + v_2$.
  (COMES FROM $(v_1 + v_2) \cdot w = v_1 \cdot w + v_2 \cdot w$)

HOW DO YOU CHECK IF SOMETHING IS IN $W^\perp$? THEOREM: IF $W = \text{SPAN}(w_1, \ldots, w_k)$ THEN v IS IN $W^\perp \iff v$ IS PERPENDICULAR TO $w_1, w_2, \ldots, w_k$.

EXAMPLE:

$$W = \text{SPAN}\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right)$$

WHAT IS $W^\perp$? CONSISTS OF $\begin{bmatrix}x\\y\\z\end{bmatrix}$ SATISFYING

$$0 = \begin{bmatrix}1\\1\\1\end{bmatrix}\cdot\begin{bmatrix}x\\y\\z\end{bmatrix} = x+y+z.$$

$$W^\perp = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x+y+z=0\right\}$$

$$= \text{SPAN}\left(\begin{bmatrix}1\\-1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\-1\end{bmatrix}\right)$$

WHAT IS $(W^\perp)^\perp$? JUST W!

$$(W^\perp)^\perp = \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : \begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}1\\-1\\0\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}\cdot\begin{bmatrix}1\\0\\-1\end{bmatrix} = 0\right\}$$

$$= \left\{\begin{bmatrix}x\\y\\z\end{bmatrix} : x-y=0,\ x-z=0\right\} = \left\{\begin{bmatrix}x\\x\\x\end{bmatrix}\right\} = W.$$

ANOTHER WAY TO UNDERSTAND THIS:

$~~W = COL~~$ $A = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ 3x1 MATRIX

$$\boxed{W = COL(A), \qquad W^{\perp} = NUL(A^T)}$$

$A^T = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$ 1x3 MATRIX.

THEOREM: FOR ANY MATRIX A,

$$\boxed{COL(A)^{\perp} = NUL(A^T)}$$

LETS US COMPUTE ANY $W^{\perp}$

SAY $W = \boxed{SPAN\left(\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix} \right)} = COL\left(\begin{bmatrix} 1 & 1 \\ 1 & -1 \\ 1 & 1 \\ 1 & -1 \end{bmatrix} \right)$

$W^{\perp} = NUL\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \right)$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -2 & 0 & -2 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$

x_3, x_4 FREE

SO $W^{\perp} = \text{SPAN}\left(\begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right).$

<u>THEOREM</u>: IF W IS A SUBSPACE OF $\mathbb{R}^n$,

$$\dim W^{\perp} = n - \dim W$$

$$(n = \dim W + \dim W^{\perp}).$$