LECTURE 13 - 2/16/17

STARTING WITH FEEDBACK FORMS.

LAST COUPLE OF CLASSES, HAD THREE IMPORTANT DEFINITIONS:

- SUBSPACE OF $\mathbb{R}^n$ (SOMETHING GENERALIZING LINES/PLANES).
- BASIS OF A SUBSPACE (LIST OF VECTORS $v_1, \ldots, v_r$ IN THE SUBSPACE H SUCH THAT
  1. IT'S LINEARLY INDEPENDENT
  2. IT SPANS ($\text{SPAN}(v_1, \ldots, v_r) = H$)
- DIMENSION OF A SUBSPACE: $\text{DIM}(H)$ = SIZE OF A BASIS. (ALL BASES HAVE SAME SIZE).

LOTS OF HEAVY CONCEPTS HERE! FOR NOW FOCUSED ON TWO PARTICULAR SITUATIONS: FOR A $m \times n$ MATRIX A,

- COLUMN SPACE = $\text{COL}(A)$ = SPAN (COLUMNS OF A).
  (= THE IMAGE/RANGE OF THE FUNCTION $T(v) = Av$).
  SUBSPACE OF $\mathbb{R}^m$
- NULL SPACE = $\text{NUL}(A)$ = ALL x IN $\mathbb{R}^n$ SUCH THAT $Ax = 0$.
  SUBSPACE OF $\mathbb{R}^n$.

DIMENSION OF $\text{COL}(A)$, $\text{DIM}(\text{COL}(A))$ IS CALLED THE RANK OF THE MATRIX.

$\text{DIM}(\text{NUL}(A))$ IS ALSO IMPORTANT - WE WON'T GIVE IT A SPECIAL NAME. (SOMETIMES CALLED NULLITY).

EXAMPLE FROM LAST TIME: $4 \times \underline{5}$ MATRIX

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 3 \\ 2 & -1 & -5 & 1 & 2 \\ -3 & 1 & 7 & -6 & -8 \\ 0 & -2 & -2 & 5 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -2 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

LAST TIME COMPUTED: 1ST, 2ND, 4TH COLS.

$$\mathrm{COL}(A) = \mathrm{SPAN}\left(\underbrace{\begin{bmatrix} 1 \\ 2 \\ -3 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -6 \\ 5 \end{bmatrix}}_{\text{BASIS}} \right)$$

(NOTE: TOOK COLUMNS FROM ORIGINAL MATRIX WITH PIVOTS IN RREF). THESE GIVE A BASIS OF COLUMN SPACE).

$$\mathrm{NUL}(A) = \mathrm{SPAN}\left(\underbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}}_{\text{BASIS}} \right)$$

GET THIS BECAUSE NUL(A) = SOLUTION SPACE TO $Ax=0$; SOLVED EQUATION IN PARAMETRIC VECTOR FORM

~~SOLUTION~~

GOT SOLUTION SPACE WAS

$$x_3 \begin{bmatrix} 2 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -1 \\ -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}$$

(ONE VECTOR PER FREE VARIABLE I.E. ONE PER NON-PIVOT COLUMNS)

SOLVING $Ax=0$ IN PARAMETRIC VECTOR FORM, GET ONE VECTOR PER FREE VARIABLE; THESE GIVE A BASIS FOR NUL(A).

DIM (COL(A)) = 3 (RANK(A) = 3).
DIM (NUL(A)) = 2.

NOTE: $3+2=5$

RANK THEOREM: FOR AN $m \times n$ MATRIX A,

$$\boxed{\text{RANK}(A) + \text{DIM}(\text{NUL}(A)) = n.}$$

OF COLUMNS OF A.

WHY IS THIS TRUE?

~~PIVOT COLUMNS~~

- # PIVOT COLUMNS = DIM(COL(A)) = RANK(A).
- # NON-PIVOT COLUMNS = DIM(NUL(A)).

OTHER EXAMPLES: GEOMETRIC LINEAR TRANSFORMATIONS $\mathbb{R}^2 \to \mathbb{R}^2$.

- $T(x) = Ax$ IS ROTATION BY SOME ANGLE. ($\mathbb{R}^2 \to \mathbb{R}^2$, SO A IS 2×2).

$$\boxed{\begin{array}{l} \text{RANK}(A) = 2 \\ \text{DIM}(\text{NUL}(A)) = 0. \end{array}}$$

$\leftarrow$ $\text{COL}(A) = \mathbb{R}^2$ (CAN GET ANYTHING IN IMAGE OF A ROTATION)

$\leftarrow$ $\text{NUL}(A) = \{0\}$.

- $T(x) = Ax$ = PROJECTION TO X-AXIS

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

$$\text{COL}(A) = \text{SPAN}\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right)$$

$$\text{NUL}(A) = \text{SPAN}\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right).$$

$$\boxed{\begin{array}{l} \text{RANK}(A) = 1 \\ \text{DIM}(\text{NUL}(A)) = 1. \end{array}}$$

- $T(x) = 0$, $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$\boxed{\begin{array}{l} \text{RANK}(A) = 0 \\ \text{DIM}(\text{NUL}(A)) = 2. \end{array}}$$

3 POSSIBLE EXAMPLES FOR 2×2 MATRICES!

ANOTHER EXAMPLE: $\mathbb{R}^3 \to \mathbb{R}^3$ (3x3 MATRIX)

$$T(x) = Ax \qquad A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\text{COL}(A) = \text{SPAN}\left(\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right) \leftarrow \text{PLANE IN } \mathbb{R}^3$$

RANK(A) = 2.

DIM (NUL(A)) = 1

$$\text{NUL}(A) = \text{SPAN}\left(\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right) \leftarrow \text{LINE IN } \mathbb{R}^3.$$

(NOTE: THIS PLANE & THIS LINE DON'T NECESSARILY HAVE ANY GEOMETRIC RELATION).

CAN ADD ON NEW PARTS TO INVERTIBLE MATRIX THEOREM:

~~A IS~~ I

THE FOLLOWING ARE EQUIVALENT FOR AN $n \times n$ MATRIX A:

(1) A IS INVERTIBLE

(2)

⋮

(13)

(14) COLUMNS OF A ARE A BASIS FOR $\mathbb{R}^n$.

(15) $\text{COL}(A) = \mathbb{R}^n$.

(16) $\text{RANK}(A) = \text{DIM}(\text{COL}(A)) = n$.

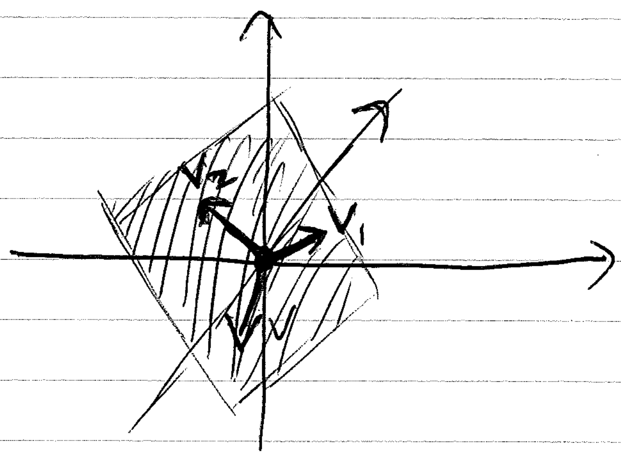
(17) $\text{DIM}(\text{NUL}(A)) = 0$.

(18) $\text{NUL}(A) = \{0\}$.

COORDINATES: (LINKED TO NOTES ON WEBPAGE)

IDEA: HOW DO I THINK ABOUT A PLANE IN $\mathbb{R}^3$?
NOT LITERALLY $\mathbb{R}^2$, BUT "LOOKS LIKE IT".

MAKE THIS PRECISE: IDENTIFY $\mathbb{R}^2$ WITH THE PLANE IN $\mathbb{R}^3$ BY CHOOSING A COORDINATE SYSTEM.



PICK A BASIS $\{v_1, v_2\}$ OF THE PLANE.

BECAUSE IT'S A BASIS ANY v IN THE PLANE CAN BE WRITTEN IN TERMS OF v_1, v_2:

$$v = x v_1 + y v_2.$$

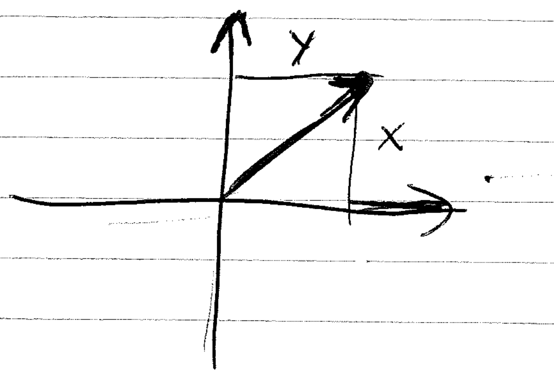
WANT TO IDENTIFY v WITH ITS "COORDINATES" x AND y (IN TERMS OF v_1, v_2).

NOTATION: IF H IS A SUBSPACE OF $\mathbb{R}^n$, AND $\mathcal{B} = \{v_1, \dots, v_m\}$ IS A BASIS, WRITE:

$$[v]_{\mathcal{B}} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} \iff v = x_1 v_1 + x_2 v_2 + \dots + x_m v_m.$$

COORDINATE VECTOR FOR v IN TERMS OF $\mathcal{B}$

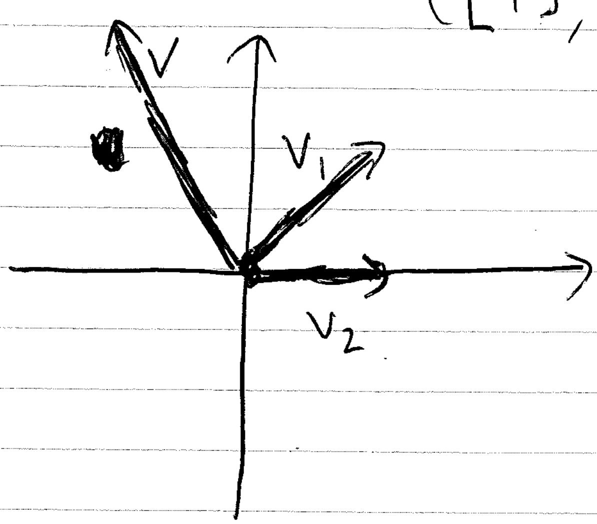
REALLY DOING HERE: CHOOSING A <u>REFERENCE FRAME</u> (<u>COORDINATE SYSTEM</u>) FOR OUR SUBSPACE H.
SORT OF LIKE WE CHOOSE X-AXIS, Y-AXIS IN $\mathbb{R}^2$



REMARKS/WARNINGS:

- ALLOWING <u>ANY</u> BASIS $\mathcal{B}$ - DOESN'T HAVE TO BE PERPENDICULAR!
- ALLOW $H = \mathbb{R}^n$ - CAN TALK ABOUT NEW COORDINATE SYSTEMS IN $\mathbb{R}^n$.

EXAMPLE: $\mathcal{B} = \left\{ \begin{bmatrix}1\\1\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix} \right\} = \{v_1, v_2\}$ BASIS FOR $\mathbb{R}^2$.



$$\boxed{V = \begin{bmatrix}-1\\2\end{bmatrix}} = \overset{2}{}\begin{bmatrix}1\\1\end{bmatrix} + \overset{-3}{}\begin{bmatrix}1\\0\end{bmatrix}$$

$$= x_1 v_1 + x_2 v_2$$

$$\boxed{[V]_B = \begin{bmatrix}2\\-3\end{bmatrix}}$$

WANT TO THINK OF v AS BEING OUR "ACTUAL VECTOR" GEOMETRICALLY, AND $[v]_B$ AS BEING AN "ALGEBRAIC REPRESENTATION" IN TERMS OF A COORDINATE SYSTEM.

WANT TO THINK OF v AND $[v]_B$ AS BEING DIFFERENT SORTS OF OBJECTS - LIVING IN "DIFFERENT PLACES". (EVEN THOUGH LITERALLY BOTH IN $\mathbb{R}^2$).

WANT TO THINK OF THEM AS <u>REPRESENTING</u> THE SAME THING (FROM DIFFERENT PERSPECTIVES) - EVEN THOUGH THEY'RE NOT EQUAL AS ELEMENTS OF $\mathbb{R}^2$.

SO THIS REALLY DEPENDS ON <u>CONTEXT</u> OF WHAT WE WANT EACH THING TO MEAN.

(SO: $v + [v]_B$ LITERALLY MAKES SENSE, BUT DOESN'T ACTUALLY <u>MEAN</u> ANYTHING IN CONTEXT)

SINCE v, $[v]_B$ LITERALLY ARE VECTORS IN $\mathbb{R}^2$,

$T(v) = [v]_B$ IS A FUNCTION $\mathbb{R}^2 \to \mathbb{R}^2$.
CAN CHECK IT'S LINEAR.

SO THERE'S A MATRIX P_B SATISFYING $\boxed{P_B v = [v]_B}$

<u>CHANGE-OF-BASIS MATRIX</u>

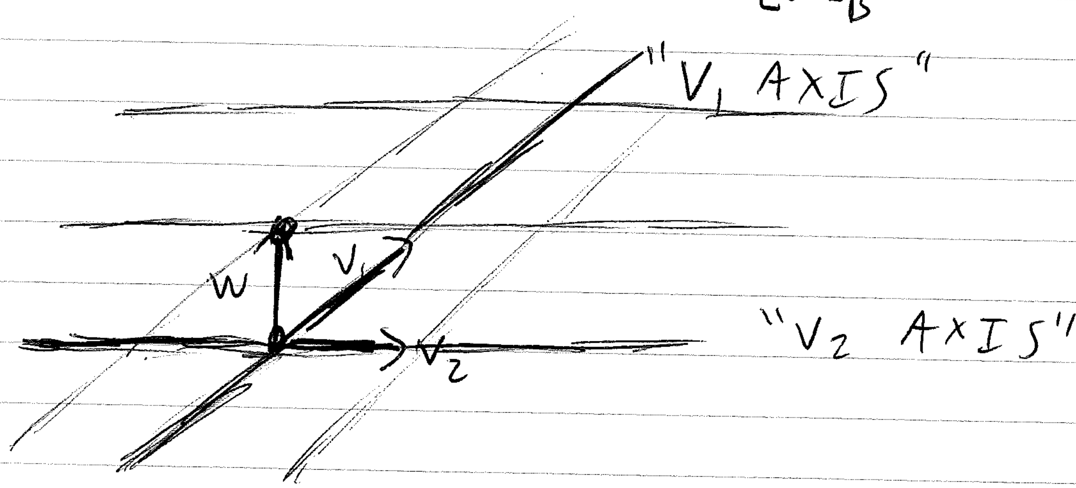
FOR $B = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ AS BEFORE:

$P_B = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ $\cancel{P_B v}$

FOR $v = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$: $P_B v = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -3 \end{bmatrix} = [v]_B$.

SIMILARLY:

$$P_B \underbrace{\begin{bmatrix} 0 \\ 1 \end{bmatrix}}_{w} = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ -1 \end{bmatrix}}_{[w]_B} = \cancel{0}$$



W IS AT POSITION 1 FROM THE V_1 AXIS & −1 FROM V_2 AXIS.