Problem 1. [10 points] Compute the determinant of the following matrix and determine if it is invertible. (As usual, remember to show your work!)

\[
A = \begin{bmatrix}
  1 & 0 & 2 & 0 \\
  0 & 3 & -1 & 1 \\
  1 & 1 & 2 & 7 \\
  0 & 2 & -2 & 2
\end{bmatrix}.
\]

One way to proceed is by row-reduction, keeping track of how the row operations change the matrix:

\[
\begin{align*}
\det \begin{bmatrix}
  1 & 0 & 2 & 0 \\
  0 & 3 & -1 & 1 \\
  1 & 1 & 2 & 7 \\
  0 & 2 & -2 & 2
\end{bmatrix}
&= 2 \det \begin{bmatrix}
  1 & 0 & 2 & 0 \\
  0 & 3 & -1 & 1 \\
  0 & 1 & 0 & 7 \\
  0 & 1 & -1 & 1
\end{bmatrix} \\
&= 2 \det \begin{bmatrix}
  1 & 0 & 2 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 1 & 1 & 6 \\
  0 & 1 & -1 & 1
\end{bmatrix} \\
&= 4 \det \begin{bmatrix}
  1 & 0 & 2 & 0 \\
  0 & 1 & 0 & 0 \\
  0 & 0 & 1 & 6 \\
  0 & 0 & 0 & 7
\end{bmatrix} \\
&= 28.
\end{align*}
\]
Problem 2. [10 points] Find the inverse of the following matrix.

\[
A = \begin{bmatrix}
1 & -2 & 1 \\
-2 & 5 & -3 \\
1 & -1 & 1
\end{bmatrix}.
\]

The standard way to do this is to row-reduce \([A|I_3]\) to get \([I_3|A^{-1}]\), which proceeds as

\[
\begin{bmatrix}
1 & -2 & 1 & | & 1 & 0 & 0 \\
-2 & 5 & -3 & | & 0 & 1 & 0 \\
1 & -1 & 1 & | & 0 & 0 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & -2 & 1 & | & 1 & 0 & 0 \\
0 & 1 & -1 & | & 2 & 1 & 0 \\
0 & 0 & 1 & | & -3 & -1 & 1
\end{bmatrix} \sim \begin{bmatrix}
1 & 0 & 0 & | & 2 & 1 & 1 \\
0 & 1 & 0 & | & -1 & 0 & 1 \\
0 & 0 & 1 & | & -3 & -1 & 1
\end{bmatrix}
\]

(combining a few steps). So we get

\[
A^{-1} = \begin{bmatrix}
2 & 1 & 1 \\
-1 & 0 & 1 \\
-3 & -1 & 1
\end{bmatrix}.
\]
Problem 3. [15 points] Consider the following matrix.

\[ A = \begin{bmatrix} 4 & 2 & 1 & 4 & 1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}. \]

(a) [7 points] Find a basis for the column space of \( A \).
(b) [8 points] Find a basis for the null space of \( A \).

Once again, the first step to solving both parts of this is by putting \( A \) into RREF:

\[ \begin{bmatrix} 4 & 2 & 1 & 4 & 1 \\ 0 & 0 & -1 & 2 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & \frac{1}{2} & 0 & \frac{3}{2} & 0 \\ 0 & 0 & 1 & -2 & 0 \\ 0 & 0 & 1 & -2 & 1 \end{bmatrix}. \]

At this point we can see there are pivots in the first, third, and fifth columns, so the corresponding columns of the original matrix give a basis for the span:

\[ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}. \]

(Of course this means \( \text{Col}(A) = \mathbb{R}^3 \) and any basis of \( \mathbb{R}^3 \) is a basis of the column space, including \( \vec{e}_1, \vec{e}_2, \vec{e}_3 \) but you need to justify this rather than just reading off these columns from the RREF matrix!). For the null space, we get a free variable from the second and fourth columns, and writing the solution in parametric vector form we can find a basis of

\[ \begin{bmatrix} -1/2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3/2 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}. \]
Problem 4. [10 points] The following five questions are worth 2 points each. For each you should include a brief explanation of the reasoning for your answer.

1. Write down a $3 \times 3$ matrix with null space spanned by the vector $\begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}$. (Make sure to justify why your null space isn’t any bigger!)

2. Write down a $2 \times 2$ matrix with column space spanned by the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and null space spanned by the vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

3. If $A, B$ are $n \times n$ matrices such that $\det(A) = 9$ and $\det(B) = 3$, compute $\det(AB^{-1})$.

4. If $T : \mathbb{R}^5 \rightarrow \mathbb{R}^4$ has rank 2, what is the dimension of its null space?

5. For what values of $x$ is the following matrix invertible?

\[
\begin{bmatrix}
x & x & 2 \\
0 & x & x \\
x & 0 & x \\
\end{bmatrix}
\]

(1) To guarantee that our null space is only 1-dimensional, we need to make sure our matrix has two pivots, and to guarantee that this vector is in the null space (and thus spans) we need to make sure the third column is equal to $-3$ times the first plus $-2$ times the second. The following RREF matrix works:

\[
\begin{bmatrix}
1 & 0 & -3 \\
0 & 1 & -2 \\
0 & 0 & 0 \\
\end{bmatrix}
\]

(2) To give the column space we want we need to make both columns a multiple of the vector, and to make the null space contain the vector we want we need to make sure the first column minus the second column equals 0, i.e. they’re equal. So we can take

\[
\begin{bmatrix}
1 & 1 \\
1 & 1 \\
\end{bmatrix}
\]

(3) We have $\det(AB^{-1}) = \det(A) \cdot \det(B)^{-1} = 9 \cdot \frac{1}{3} = 3$.

(4) The rank theorem tells us $\dim \text{Nul}(A) = n - \text{rk}(A) = 5 - 2 = 3$.

(5) Applying the diagonal method we can compute this determinant to be $2x^3 - 2x^2 = 2x^2(1 - x)$. This is nonzero (and thus the matrix is invertible) exactly when $x \neq 0, 1$. 

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Problem 5. [10 points] For each of the following state whether the statement is TRUE or FALSE (no justification is necessary).

1. It is possible for $\text{Col}(A) = \text{Nul}(A)$ when $A$ is a $3 \times 3$ matrix.

2. If $A, B$ are $n \times n$ matrices with $A$ invertible, it is always true that $\det(ABA^{-1}) = \det(B)$.

3. If $A$ is an $n \times n$ matrix and $xA$ is the scalar multiple of the entire matrix $A$ by the scalar $x$, it is always true that $\det(xA) = x \det(A)$.

4. If $A$ is a $4 \times 4$ matrix such that the homogeneous system of linear equations $A\vec{x} = \vec{0}$ has only the trivial solution, then it is onto.

5. If $A$ is a $2 \times 2$ matrix with integer entries and satisfying $\det(A) = -1$, then $A$ is invertible and $A^{-1}$ also has integer entries and satisfies $\det(A^{-1}) = -1$.

(1) **False.** Since $n = 3$ is the sum of the dimensions of these two spaces, the dimensions (and thus the spaces) can’t ever be equal.

(2) **True.** This follows because $\det(ABA^{-1}) = \det(A) \det(B) \det(A)^{-1}$, and since multiplication of numbers is commutative we can switch the order and cancel out $\det(A)$ with $\det(A^{-1})$.

(3) **False.** The correct formula is $\det(xA) = x^n \det(A)$, since we have to remove the multiple of $x$ one row at a time using our rules for manipulating row operations with determinants.

(4) **True.** This is a consequence of the invertible matrix theorem.

(5) **True.** We know $\det(A^{-1}) = \det(A)^{-1} = -1$, and using our formula for $2 \times 2$ inverses we have

\[
A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \implies A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} -d & b \\ c & -a \end{bmatrix};
\]

since $a, b, c, d$ are integers by assumption so are all of the entries in $A^{-1}$.