Problem 1. [15 points] Solve the matrix equation $Ax = b$ for

$$A = \begin{bmatrix} 2 & 4 & 6 & 8 \\ 3 & 6 & 10 & 13 \end{bmatrix}, \quad b = \begin{bmatrix} 6 \\ 10 \end{bmatrix}.$$  

Write your answer in parametric vector form.

Solution. Row-reducing the augmented matrix $[A|b]$ gets us to

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \end{bmatrix}.$$  

From this we can read off that $x_2$ and $x_4$ are free variables, and the general solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - x_4 \\ 0 \\ -x_4 + 1 \\ x_4 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -1 \\ 0 \\ 0 \\ -1 \end{bmatrix},$$

with this latter expression in parametric vector form.

Problem 2. [15 points] For the following linear transformations $\mathbb{R}^2 \to \mathbb{R}^2$ that are described geometrically, determine what each does to the standard basis vectors $e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and then find the matrix of the transformation.

(a) $S : \mathbb{R}^2 \to \mathbb{R}^2$ given by first reflecting across the line $y = x$, and then reflecting across the $x$-axis.

Solution. Reflecting $e_1$ across the line $y = x$ gives $e_2$, and reflecting across the $x$-axis gives $-e_2$, so $S(e_1) = -e_2$. Reflecting $e_2$ across the line $y = x$ gives $e_1$ and reflecting that across the $x$-axis gives $e_1$ again, so $S(e_2) = e_1$. Thus our matrix is

$$A = \begin{bmatrix} S(e_1) & S(e_2) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$  

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by first applying the shear transformation

$$\begin{bmatrix} x \\ y \end{bmatrix} \mapsto \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ x + y \end{bmatrix}$$

and then rotating by 90 degrees counterclockwise.

Solution. Here the shear takes $e_1$ to $e_1$ and rotating takes it to $e_2$, so $T(e_1) = e_2$. Similarly the shear takes $e_2$ to $e_1 + e_2$ and rotating takes that to $-e_1 + e_2$, so $T(e_2) = -e_1 + e_2$. So the matrix is

$$A = \begin{bmatrix} T(e_1) & T(e_2) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 1 \end{bmatrix}.$$  

Problem 3. [15 points] Solve the inhomogeneous linear system $x_1v_1 + x_2v_2 + x_3v_3 = w$ for the following vectors:

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 3 \\ 0 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}, \quad w = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$  

Does this tell you that $w$ is or is not in the span of $v_1, v_2, v_3$?
Solution. Beginning to row-reduce the augmented matrix $[v_1 \ v_2 \ v_3 | w]$ leads us to something like

$$
\begin{bmatrix}
-1 & 0 & 2 & -1 \\
0 & 3 & 6 & 0 \\
0 & 0 & 0 & 1 \\
\end{bmatrix},
$$

at which point we can see the system is inconsistent, i.e. has no solutions. Since there are no solutions of our attempt to write $w$ as a linear combination of $v_1, v_2, v_3$, we conclude $w$ is not in the span.

Problem 4. [20 points] Each of the four parts of this problem describes a function. For each one, decide if the function is linear or not linear. If it is linear, write down the matrix corresponding to it. If it’s not linear, find two specific vectors $v, w$ such that you can show $T(v + w) \neq T(v) + T(w)$.

(a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by

$$
T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} 3x + 2y + z \\ 3x + 3y + 2z + 1 \end{bmatrix}.
$$

Solution. This is not linear, because $T(0) \neq 0$, and thus $T(0) + T(0) \neq T(0)$.

(b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ where $T(v)$ is given by rotating $v$ by 180 degrees and multiplying the length by 3.

Solution. This is linear, as it comes from some geometric things we said were linear - scaling and rotating. In particular rotating by 180 degrees is the same as multiplying by the scalar $-1$, so really $T(v) = -3v$ and the matrix is thus

$$
\begin{bmatrix}
-3 & 0 \\
0 & -3 \\
\end{bmatrix}.
$$

(c) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by defining $T(v)$ to have the same direction as $v$ but length 1 longer than the length of $T(v)$. (And setting $T(0) = 0$).

Solution. This is not linear. Looking at the vector $e_1$ of length 1, we see $T$ takes it to something of length 2 in the same direction, i.e. $2e_1$. Also $T$ takes $2e_1$ to something of length $3 = 2 + 1$ in the same direction, i.e. $3e_1$. Thus

$$
T(e_1) + T(e_1) = 4e_1 \neq 3e_1 = T(e_1 + e_1).
$$

(d) $T : \mathbb{R}^3 \to \mathbb{R}^3$ given by

$$
T \left( \begin{bmatrix} x \\ y \\ z \end{bmatrix} \right) = \begin{bmatrix} x + y \\ y + z \\ z + x \end{bmatrix}.
$$

Solution. This is linear, as we can directly see it is the matrix transformation for the following matrix

$$
\begin{bmatrix}
1 & 1 & 0 \\
0 & 1 & 1 \\
1 & 0 & 1 \\
\end{bmatrix}.
$$

Problem 5. [15 points] Suppose a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ satisfies

$$
T \left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix}, \quad T \left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix}.
$$
Solve for the entries of the $3 \times 2$ matrix

$$A = \begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$$

such that $T(v) = Av$.

**Solution.** One way to do this is to simply write out the product of $A$ times each of the two vectors we’re given and conclude

$$\begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix} = T\left( \begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = A \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2a + b \\ 2c + d \\ 2e + f \end{bmatrix}$$

and

$$\begin{bmatrix} -2 \\ 4 \\ 0 \end{bmatrix} = T\left( \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) = A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} a - b \\ c - d \\ e - f \end{bmatrix}.$$  

This gives us a system of six equations in six variables (really three separate systems of two equations in two variables!) which we can solve for $a, b, c, d, e, f$ and find

$$A = \begin{bmatrix} 1 & 3 \\ 3 & -1 \\ 2 & 2 \end{bmatrix}.$$  

**Problem 6.** [20 points] For which real numbers $h$ is the following list of vectors **linearly dependent**, and for which $h$ is the list **linearly independent**?

$$v_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 3 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 3 \\ -2 \\ h \\ 5 \end{bmatrix}$$

**Solution.** We row-reduce the augmented matrix for the homogeneous system $x_1v_1 + x_2v_2 + x_3v_3 = 0$ and find we get to

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -h - 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$  

At this point we can see that if $h \neq -1$, there is a pivot in every column so there are no free variables in our dependence relation, which means the system is **linearly independent** when $h \neq -1$. If $h = -1$, however, we do get a free variable for the third column and the system is **linearly dependent**.