INVERTIBLE MATRICES.

AN $n \times n$ MATRIX $A$ IS INVERTIBLE, IF THERE IS AN $n \times n$ MATRIX $A^{-1}$ (ITS INVERSE) SATISFYING:

$$AA^{-1} = A^{-1}A = I_n.$$ 

NOTE: $A^{-1}$ IS UNIQUE (IF IT EXISTS) (THE INVERSE).

WHY? SUPPOSE 2 DIFFERENT INVERSES $B, C$.

$$AB = BA = I_n \quad AC = CA = I_n.$$ 

$$B = B \cdot I_n = B(AC) = (BA)C = I_n \cdot C = C$$

HOW TO COMPUTE:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{(IF } ad-bc \neq 0)\text{.}$$

IN GENERAL: ROW-REDUCE $[A | I_n]$.

$$[I_n | A^{-1}] \quad \text{NOT INVERTIBLE}$$

$$[\text{NOT } I_n | -] \quad \text{NOT IN INVERTIBLE}.$$
WHAT CAN WE DO WITH $A^{-1}$?

IF $A$ IS INVERTIBLE, $A\vec{x} = \vec{b}$ ALWAYS HAS A UNIQUE SOLUTION, $\vec{x} = A^{-1}\vec{b}$

($A\vec{x} = \vec{b} \iff A^{-1}A\vec{x} = A^{-1}\vec{b} \iff \vec{x} = A^{-1}\vec{b}$)

EX: SOLVE $\begin{array}{c}
    x + 2y = 4 \\
    3x + 5y = 7 \\
\end{array}$

$A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix}$

$\vec{b} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$, $\vec{x} = A^{-1}\vec{b} = \begin{bmatrix} -5 & 2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} -6 \\ 5 \end{bmatrix}$

SOLUTION: $x = -6$, $y = 5$.

DOWNside: FINDING $A^{-1}$ IS HARDER THAN JUST SOLVING $A\vec{x} = \vec{b}$. (USUALLY).
THEORETICAL SIDE: LOTS OF THINGS = "A IS INVERTIBLE"

INVERTIBLE MATRIX THEOREM: SQUARE MATRICES ONLY

Let A be an $m \times n$ matrix. Then the following are equivalent.

1. A is invertible.
2. A can be row-reduced to In.
3. A has pivots in every column.
4. A has pivots in every row.
5. The equation $A \mathbf{x} = \mathbf{0}$ has only the trivial solution.
6. The columns of A are linearly independent.
7. The linear transformation $T(\mathbf{x}) = A \mathbf{x}$ is one-to-one.
8. The equation $A \mathbf{x} = \mathbf{0}$ has at least one solution for all $\mathbf{b}$. 
9. The columns of A span $\mathbb{R}^m$.
10. The linear transformation $T(\mathbf{x}) = A \mathbf{x}$ is onto.
11. There is an $m \times n$ matrix C such that $CA = In$.
12. There is an $n \times m$ matrix D such that $AD = In$.
13. The matrix $A^T$ is invertible.
EX: \[ A = \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \]

1. INVERTIBLE

2. \[ \begin{bmatrix} 1 & 2 \\ 3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \]

3. & 4. YES

5. \[ x + 2y = 0 \] \implies x = y = 0.
   \[ 3x + 5y = 0 \] \text{ ONLY sol.}

6. COLUMNS \[ \begin{bmatrix} 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 5 \end{bmatrix} \]
   LIN INDEPENDENT.

B = \[ \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \]

NOT INVERTIBLE.

\[ \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix} \]

NO.

x + 2y = 0 solutions like x = 2, y = 1

3x + 6y = 0

\[ \begin{bmatrix} 1 & 2 \\ 3 & 6 \end{bmatrix} \]

LINEARLY DEPENDENT.
WHY ARE THESE ALL THE SAME?

1 ⇔ 2: \([A \mid I_n]\) row reduces to \([I_n \mid A^{-1}]\).

2 ⇔ 3,4: LOOK AT RREF's FOR SQUARE MATRIX...

3, 5, 6, 7: SAID THESE ARE THE SAME IN GENERAL.

4, 8, 9, 10: SAID THESE ARE THE SAME IN GENERAL.

11, 12, 13: SKIP OVER WHY THESE ARE THE SAME.

IMPORTANT TO NOTE: (11) & (12) SAY A "ONE-SIDED INVERSE" OF A SQUARE MATRIX IS A "TWO-SIDED INVERSE".
Importance of theorem: Ties together a whole bunch of concepts we have (for square matrices). We'll come back to these connections a lot.

For now: lots of ways to look at matrix & tell if it's invertible.

Ex: $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

Invertible

Use (3) or (4).

$B = \begin{bmatrix} 4 & -4 & 7 \\ 2 & -2 & 4 \\ 3 & -3 & 6 \end{bmatrix}$

Not invertible

Lin. Dependent columns.
WHAT ABOUT NON-SQUARE MATRICES?

FACT: IF A IS m x n, m ≠ n, (NOT SQUARE) THERE CANNOT BE B (n x m) WITH

AB = In, BA = In.

THERE CAN BE (MANY) MATRICES THAT WORK ON JUST ONE SIDE!

EX: 

A = \[
\begin{bmatrix}
1 & 2 & 2 \\
3 & 5 & 5
\end{bmatrix}
\]

B = \[
\begin{bmatrix}
-5 & 2 \\
3 & -1 \\
0 & 0
\end{bmatrix}
\]  

C = \[
\begin{bmatrix}
-5 & 2 \\
2 & -2 \\
1 & 1
\end{bmatrix}
\]

AB = \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] = I_2  

AC = \[
\begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix}
\] = I_2.

BA = \[
\begin{bmatrix}
-5 & 2 \\
3 & -1 \\
0 & 0
\end{bmatrix}
\] \[
\begin{bmatrix}
1 & 2 & 2 \\
3 & 5 & 5
\end{bmatrix}
\] = \[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 0
\end{bmatrix}
\] ≠ I_3.

CA = \[
\begin{bmatrix}
-5 & 2 \\
2 & -2 \\
1 & 1
\end{bmatrix}
\] \[
\begin{bmatrix}
1 & 2 & 2 \\
3 & 5 & 5
\end{bmatrix}
\] = \[
\begin{bmatrix}
1 & 0 & 0 \\
-4 & -6 & -6 \\
4 & 7 & 7
\end{bmatrix}
\] ≠ I_3.
BACK TO LINEAR TRANSFORMATIONS & FUNCTIONS.

IF \( A \) IS A SQUARE MATRIX, \( T : \mathbb{R}^n \to \mathbb{R}^n \)
\[ (n \times n) \]
\[ T(x) = Ax. \]

INVERTIBLE MATRIX THEOREM SAYS
IF \( A \) IS INVERTIBLE, THEN
\( T \) IS ONE-TO-ONE AND ONTO.

FOR ANY \( \hat{y} \), THERE IS A UNIQUE \( x \)
WITH \( T(x) = \hat{y} \).

\( \Rightarrow \) WE CAN DEFINE AN INVERSE FUNCTION

\[ T^{-1} : \mathbb{R}^n \to \mathbb{R}^n \] \( \text{BY} \)
\[ T^{-1}(\hat{y}) = \hat{x} \]
WHERE \( \hat{x} \) IS THE UNIQUE VECTOR WITH \( T(\hat{x}) = \hat{y} \).

(SO \( T^{-1} \) "UNDOES" \( T \).

SATISFIES
\[ T^{-1}(T(\hat{x})) = \hat{x}, \quad T(T^{-1}(\hat{y})) = \hat{y}. \]

THEOREM: IF \( T : \mathbb{R}^n \to \mathbb{R}^n \) IS A LINEAR TRANSFORMATION
& \( A \) IS ITS MATRIX, \( T \) IS AN INVERTIBLE FUNCTION
(AND: \( T^{-1}(\hat{y}) = A^{-1}\hat{y} \))