Gaussian Measures

In this lecture the basic facts about Gaussian measures are introduced, but with a slant towards their role in theoretical physics where they serve as the underpinning for quantum field theory. Therefore the connection with graphs, Hermite polynomials, etc is included.

Gaussian Lattice Fields

$\Lambda \subset \mathbb{Z}^d, \qquad |\Lambda| < \infty$

$\Phi = (\phi_x, x \in \Lambda)$

$A = (A_{xy}, x, y \in \Lambda)$

A is symmetric with positive eigenvalues.

$(\phi, A\phi) > 0$ if $\phi \neq 0$. $\qquad$ A is said to be positive definite

Define prob. measure on $\mathbb{R}^\Lambda$ by *

$$d\mu_C(\phi) = \frac{1}{N} e^{-\frac{1}{2}(\phi, A\phi)} d^\Lambda\phi$$

$C = A^{-1}$

Then

(1) $\displaystyle\int d\mu_C(\phi)\, e^{(f,\phi)} = e^{\frac{1}{2}(f, Cf)} \qquad f \in \mathbb{R}^\Lambda$

(2) $\displaystyle\int d\mu_C\, \phi_a \phi_b = C_{ab}$

(3) $N = (2\pi)^{|\Lambda|/2} \det^{-1/2} A$

Lemma 1 Given a $\Lambda \times \Lambda$ positive-def. matrix C there exists a unique prob. measure s.t. (1) holds and it is $d\mu_C$.

* Gaussian measures are parameterised by C rather than A because the marginals are Gaussian with the same C (restricted to a submatrix).

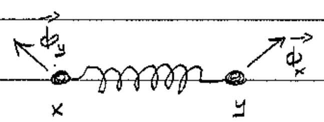
Defn 2 The massless free field is the case $A = -\Delta_\Lambda$.

The free field with mass m is the case $A = m^2 - \Delta_\Lambda$

Discussion If $\vec{\phi} : \Lambda \to \mathbb{R}^d$ is vector-valued

$$\tfrac{1}{2}\left(\vec{\phi}, -\Delta_\Lambda \vec{\phi}\right) = \tfrac{1}{2}\sum_{xy \in E} (\vec{\phi}_x - \vec{\phi}_y)^2 \qquad \phi_x = 0 \quad x \notin \Lambda$$

is the energy of all the springs in a bedspring.



$$\tfrac{1}{2}\left(\vec{\phi}_x - \vec{\phi}_y\right)\cdot\left(\vec{\phi}_x - \vec{\phi}_y\right)$$

The frame is the Dirichlet b.c.

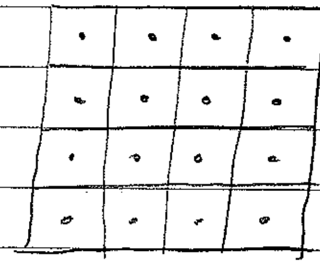
Alternatively this is a model for sound waves in a crystal.

Question 2′ For the bedspring, does ϕ_0 remember the Dirichlet b.c. as $\Lambda \nearrow \mathbb{Z}^d$?

$$\langle \phi_0 \rangle_\Lambda = 0 \quad \text{but} \quad \langle \phi_0^2 \rangle_\Lambda \;? \quad \text{as } \Lambda \nearrow \mathbb{Z}^d$$

Example 3

$$Z = \sum_{\underline{n} \in \{0,1\}^{\Lambda}} z^{\underline{n}} \, e^{\frac{1}{2} \sum_{x,y \in \Lambda} n_x v_{xy} n_y}$$



If v_{xy} is pos. det.

$$Z = \sum_{\underline{n}} z^{\underline{n}} \int d\mu_v(\phi) \, e^{\sum_{x \in \Lambda} \phi_x n_x}$$

Each box has 0, 1 particle

$$= \int d\mu_v(\phi) \sum_{\underline{n}} z^{\underline{n}} \, e^{\sum \phi_x n_x}$$

$$= \int d\mu_v(\phi) \prod_{x \in \Lambda} \left(1 + z e^{\phi_x}\right)$$

$$= \frac{1}{N} \int d^{\Lambda}\phi \; e^{-S(\phi)}$$

where

$$S(\phi) = \tfrac{1}{2}\left(\phi, v^{-1}\phi\right) - \sum_{x \in \Lambda} \log\left(1 + z e^{\phi_x}\right)$$

(Kac, Siergert, Stratonovich)

Example 3 (continued)

Possible choice:

$$v_{xy} = \beta m^2 (m^2 - \Delta_\Lambda)^{-1}_{xy}$$

$$S(\phi) = \frac{1}{2m^2\beta} \sum_{xy} (\phi_x - \phi_y)^2 + \frac{1}{2\beta} \sum_{x\in\Lambda} \phi_x^2 - \sum_{x\in\Lambda} \log(1+ze^{\phi})$$

Kac limit $\Lambda \nearrow \mathbb{Z}^d$ followed by $m \downarrow 0$.

You can see that $m \downarrow 0$ makes $\phi \simeq$ const. and understand intuitively why the limit is MFT, e.g.

$$\sum_{x\in\Lambda} \log(1+ze^{\phi}) \simeq |\Lambda| \log(1+ze^{\phi})$$

$$\frac{1}{2\beta} \sum_{x\in\Lambda} \phi_x^2 \simeq \frac{1}{2\beta} |\Lambda| \phi^2$$

The study of the associated measures as $m \downarrow 0$ is hard because the limit $\Lambda \nearrow \mathbb{Z}^d$ must be taken before $m \downarrow 0$.

Theorem 4 (Wick)

Let

$$\Delta_C = \frac{1}{2} \sum_{x,y \in \Lambda} C_{xy} \frac{\partial}{\partial \phi_x} \frac{\partial}{\partial \phi_y}$$

For P polynomial

$$\int d\mu_C \; P = e^{\frac{1}{2}\Delta_C} P \Big|_{\phi = 0}$$

Proof Homework. Hint. $\left(\frac{\partial}{\partial t} - \frac{1}{2}\Delta_C\right) \int d\mu_{tC}(\zeta)\, P(\zeta + \phi) = 0$.

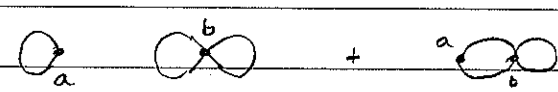
Example 5

$$\int d\mu_C \; \phi_a \phi_b = \left(1 + \frac{\Delta_C}{2} \cancel{+ \frac{1}{2!}\left(\frac{\Delta_C}{2}\right)^2 + \cdots}\right) \phi_a \phi_b \Big|_{\phi=0}$$

$$= C_{ab}$$

Example 6 (Feynman diagrams)

$$\int d\mu_C \; \frac{\phi_a^2}{2!} \frac{\phi_b^4}{4!} = \frac{1}{3!}\left(\frac{\Delta_C}{2}\right)^3 P$$

= 

$$= \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) C_{aa} \, C_{bb}^2 + \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right) C_{ab}^2 \, C_b$$

self loops 2 = # edge automorphisms

Defn 7 For polynomial P,

$$:P: \;=\; :P:_C \;=\; e^{-\frac{1}{2}\Delta_C}\, P$$

Example 8

$$:\phi_a^4: \;=\; \phi_a^4 - \tfrac{1}{2}(4)(3)\, C_{aa}\, \phi_a^2 + \tfrac{1}{2}\,\tfrac{1}{2}\,\tfrac{1}{2}\, C_{aa}^2\; 4!$$

$:\phi_x^p:$ is called the p'th Wick power. It is a monic polynomial. Also $\frac{\partial}{\partial\phi} :\phi^p: \;=\; p :\phi^{p-1}:$ follows from defn of $: \;:$

Lemma 9 If P, Q are monomial of different degree

$$\int d\mu_C \; :P: \;\; :Q: \;=\; 0$$

Remark When $|\Lambda| = 1$ this proves that $(:\phi^p:,\; p = 0, 1, \dots)$ are orthogonal polynomials on $\mathbb{R}$, so up to normalisation, they are Hermite polynomials.

Proof

$\Delta = \Delta_C$ (suppress C)

For A, B polynomials

$$e^{\frac{1}{2}\Delta} AB$$

$$= e^{\frac{1}{2}\Delta_{AA} + \Delta_{AB} + \frac{1}{2}\Delta_{BB}} AB$$

$$= e^{\Delta_{AB}} \left(e^{\frac{1}{2}\Delta_{AA}} A\right)\left(e^{\frac{1}{2}\Delta_{BB}} B\right)$$

Leibniz rule

$\frac{\partial}{\partial\phi} = \frac{\partial}{\partial\phi_A} + \frac{\partial}{\partial\phi_B}$

$A = A(\phi_A)$

$B = B(\phi_B)$

then $\phi_A = \phi_B$ after

derivs.

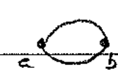
If $A = :P:$ then $e^{\frac{1}{2}\Delta_{AA}} :P: = P$

$$= e^{\Delta_{AB}} P Q$$

$$= 0 \quad \text{at } \phi = 0$$

P, Q have different degrees.

$\blacksquare$

Example 10 $\int d\mu_C \; \frac{:\phi_a^2:}{2!} \; \frac{:\phi_b^2:}{2!} =$ 

$\frac{1}{2} C_{ab}$

Why are there no self-loops?

Problems

① [In the typed notes this problem should be moved to end of Lecture 4].
Adapt Lemma 4.2, Corollary 4.4 to prove that, for A_{xy} any $\Lambda \times \Lambda$ matrix with the property

$$\frac{1}{|A_{xx}|} \sum_{y \neq x} |A_{xy}| e^{\kappa \|x-y\|} \leq c < 1, \quad x \in \Lambda,$$

A^{-1}_{xy} exists and, uniformly in Λ, decays exponentially in $\|x-y\|$.

② Prove theorem 4.

③ Answer Question 2′ for $\mathbb{Z}^2$ by proving that for f continuous with compact support

$$\langle f(\Phi_0) \rangle_\Lambda \longrightarrow 0 \quad \text{as} \quad \Lambda \nearrow \mathbb{Z}^2.$$