Lecture 5

To carry out a conditional expectation $\mathbb{E}_+$ we need

(i) $\Phi \overset{\mathcal{D}}{=} \Phi' + \zeta$ independent

(ii) ζ_x, ζ_y Gaussian, independent for $|x-y| \geq L^j/2$

In order to evaluate an expectation by carrying out a sequence of conditional expectations we need $\Phi = \sum_{j\geq 1} \zeta_j$ with ζ_{jx} ζ_{jy} as in (ii) for each $j \geq 1$. This means that the covariance $u = u_{xy} = u(x,y)$ of Φ satisfies

(1) $u = \sum_{j\geq 1} C_j$

(2) $C_j(x,y) = 0$ if $|x-y| \geq L^j/2$

<u>Open question</u> What class of $u(x,y) = u(x-y)$ have such decompositions where $C_j(x,y) = C_j(x-y)$?

In the references for these lectures are partial answers which include massive, massless lattice Gaussian fields. If, in addition to (1) and (2) we have <u>self-similarity</u>,

(3) $C_{j+1}(x,y) = L^{-2[\phi]} C_j(\frac{x}{L}, \frac{y}{L})$, $j \geq 1$,

or an asymptotic version of (3) valid in the $j \to \infty$ limit, then

$u(x,y)$? will have power law decay

$$u(x,y) \sim c\,|x-y|^{-2[\phi]}$$

as $|x-y| \to \infty$. Since $(-\Delta)^{-1}(x,y) \sim c\,|x-y|^{-(d-2)}$, $d \neq 2$, for the lattice Laplacian, the case

$$[\phi] = \frac{d-2}{2}$$

is of principal interest. We say $\frac{d-2}{2}$ is the <u>canonical dimension</u>.

In view of the open question above we do not know to what extent the converse holds, $u(x,y) \sim c\,|x-y|^{-2[\phi]} \Rightarrow$ existence of decomposition satisfying (1), (2), (3). The next Lemma answers the question when $u(x,y) = |x-y|^{-2[\phi]}$. It gives a two-sided decomposition, $j \in \mathbb{Z}$, because the negative scales capture the singularity as $|x-y| \to 0$. Of course this Lemma is not relevant for the lattice, so it is only of indirect relevance to these lectures.

In the references one can deduce that the covariance $(m^2 - \Delta)^{-1}$ for the massive Gaussian has a decomposition as in (1), (2) which obeys an approximate form of (3) for scales j such that $m\,L^j \leq 1$. For j with $mL^j \geq 1$ C_j goes to zero like $\exp(-mL^j)$ in sup norm.

Lemma Let $L > 1$, $[\phi] > 0$. There exists a C^∞ positive-definite $u(x)$ with support in $|x| \le 1/2$ s.t.

$$|x-y|^{-2[\phi]} = \sum_{j \in \mathbb{Z}} L^{-j[\phi]} u(L^{-j}(x-y))$$

for $x \neq y$.

Proof Let $v(x)$ be a smooth function of $|x|$ with support in $|x| \le 1/4$. Then $v * v$ is a positive-def. fn of $|x|$ with support in $|x| \le 1/2$. Replacing v by $v * v$ we may assume v is smooth, positive-def. with support in $|x| \le 1/2$. Let

$$f(x) = \int_0^\infty \frac{dl}{l} l^{-2[\phi]} v\left(\frac{x}{l}\right)$$

The integral is convergent near ∞ because v is bounded, and $\int^\infty \frac{dl}{l} l^{-2[\phi]}$ is convergent near ∞. It is also convergent near $l = 0$ because $v(x/l) = 0$ near $l = 0$, recalling $x \neq 0$ and v has compact support. For any $x \neq 0$ the change of variable $l = |x| l'$ proves that $f(x) = |x|^{-2[\phi]} f\left(\frac{x}{|x|}\right)$ so $f(x) = \text{const.}\ |x|^{-2[\phi]}$. By replacing v with a positive multiple of u we absorb the constant so that $f(x) = |x|^{-2[\phi]}$,

$$|x|^{-2[\phi]} = \int_0^\infty \frac{dl}{l} l^{-2[\phi]} v\left(\frac{x}{l}\right)$$

Break the range of integration into $\bigcup_j [L^{j-1}, L^j]$

$$= \sum_{j \in \mathbb{Z}} \int_{L^{j-1}}^{L^j} \frac{dl}{l} l^{-2[\phi]} v(x/l)$$

all integrals are scalings on $\int_1^L$.

which proves the Lemma holds with $u(x-y) = \int_1^L \frac{dl}{l} l^{-2[\phi]} v\left(\frac{x}{l}\right)$. ∎

RG Norms

The following (semi)norms on $\mathbb{R}^\Lambda$ are dictated by (1) multiplicativity (2) control of Taylor expansions to order p_N.

Let $\Lambda^* = \bigcup_{p\geq 0} \Lambda^p$ be the set of finite sequences of lattice points in Λ.

$$x = (x_1, \ldots, x_p) \in \Lambda^*, \qquad x = () \text{ is null sequence } (p=0).$$

$$p = p(x) \text{ is length of sequence } x$$

A Taylor expansion of $K : \mathbb{R}^\Lambda \to \mathbb{R}$ looks like

$$K(\phi+\xi) \sim \sum_{x\in\Lambda^*} \frac{1}{p!} \underbrace{K_x(\phi)}_{\frac{\partial^p K(\phi)}{\partial\phi_{x_1}\cdots\partial\phi_{x_p}}} \xi_{x_1} \cdots \xi_{x_p}$$

Given $g : \Lambda^* \to \mathbb{R}$, let

$$\langle K, g \rangle_\phi = \sum_x \frac{1}{x!} K_x(\phi)\, g_x$$

g is called a "test function". For Φ any normed space of test function, let

Drastic change in notation! From now on Φ is a space!!

$$\|K\|_{T_\phi} = \sup_{\|g\|\leq 1} \langle K, g \rangle_\phi$$

(For SI, which is a function of (ϕ, ξ) we regard ξ as a parameter)

<u>Example</u> Fix $p_N \in \mathbb{N}$.

$$\|g\|_\Phi = \sup_{\substack{X \in \Lambda^* \\ p(x) \le p_N}} \sup_{\alpha \in A} |\nabla^\alpha g(x)| \; L^{j[\phi] + j|\alpha|}$$

A: up to 2 lattice forward or backward derivs for each component of $x = (x_1, \dots, x_p)$.

Factors of L in $\|g\|_\Phi$ are such that $\left|\sum_{x:\, p \le p_N} \frac{1}{p!} K_x(0)\, \phi_{x_1} \cdots \phi_{x_p}\right|$ is $\simeq \|K\|_{T_0}$ for a typical field ϕ. The covariance of $\phi = \zeta_j + \zeta_{j+1} + \cdots$ is dominated by the covariance of C_j and $\nabla^\alpha C_j(x,x) = O\left(L^{-2j[\phi] - 2j|\alpha|}\right)$ so if we take $g_x = \phi_{x_1} \phi_{x_2} \cdots \phi_{x_p}$ then $\|g\|_\Phi$ fluctuates around 1.

<u>Lemma</u> with Φ as in example,

$$\|K_1 K_2\|_{T_\phi} \le \|K_1\|_{T_\phi} \|K_2\|_{T_\phi}$$

Proof: First show that if $\tilde{g}_z = \sum_\omega \frac{1}{p(\omega)!} G_\omega g_{z \circ \omega}$ then $\|\tilde{g}\|_\Phi \le \|G\|_T \|g\|_\Phi$

<u>Control of $\mathbb{E}_+$</u>

If, in addition we have measurable functions $G_j(X)$, $j \ge 0$, $X \in \mathcal{P}_j$,

$$\mathbb{E}_+ G_j(X) \le \alpha^{|X|_j} G_{j+1}(X)$$

$$G_j(X \cup Y) \le G_j(X)\, G_j(Y) \qquad X \cap Y = \emptyset$$

then

$$\|K(X)\|_{T_\phi} \le \epsilon\, G_j(X) \tag{ϵ}$$

$\Longrightarrow$

$$\|\mathbb{E}_+ K(X)\|_{T_{\phi'}} \le \epsilon\, \mathbb{E}_+ G_j(X) \le \epsilon\, G_{j+1}(X)\, \alpha^{|X|_j}$$

so $\| K(X) \|_j$ = best constant ϵ in (ϵ) is a basic construction for norms which factor and control $\mathbb{E}_+$. Constructions of $G_j(X)$ for ϕ^4 models and self-avoiding walk are surprisingly easy. See the bibliography. Construction of $G_j(X)$ for the anharmonic lattice is surprisingly hard. See Park City Notes.

Anharmonic Lattice

$$Z(\Lambda) = \int \underbrace{e^{-\frac{1}{2}(\nabla\phi, \nabla\phi)}}_{\text{harmonic}} \quad \underbrace{F^{\Lambda}}_{\text{anharmonic}}$$

$\Lambda \subset \mathbb{Z}^d$, periodic, integral over $\mathbb{R}^{\Lambda \setminus \{0\}}$ ($\phi(0)$ set to zero).

F_x is an even, Euclidean lattice covariant function of the gradients ($\phi_{x+e} - \phi_x$, $e \in \mathbb{Z}^d$ unit vectors). For example,

$$F_x = e^{-V\left(\sum_e (\phi_{x+e} - \phi_x)^2\right)}$$

where $V: \mathbb{R} \to \mathbb{R}$ is bounded below. Another possibility is

$$F_x = e^{z \sum_e \cos(\phi_{x+e} - \phi_x)} \qquad \text{or} \qquad 1 + z \sum_e \cos(\phi_{x+e} - \phi_x)$$

This choice equals the dipole gas (See my Park City Lectures).

$F \simeq 1$ hypothesis

$$|D^p F_x| \leq A^{*-1} h^{-p} e^{\frac{1}{2} h^{-2} (\nabla\phi)_x^2}$$

for $A \geq A_0(d)$ $\quad h \geq h_0(d, A)$

Theorem Scaling limit is same as model with $F_x = 1$ and $(\nabla\phi, \nabla\phi)$ replaced by $\gamma(\nabla\phi, \nabla\phi)$, some $\gamma = \gamma(F)$.

Scaling limit For $h \in C_0^\infty(\mathbb{R}^d)$,

$$\langle \phi, h \rangle = \sum_x \phi_x h(x)$$

$$h_\ell(x) = \ell^{-[h]} h(x/\ell) \qquad [h] = \frac{d+2}{2} \qquad (*)$$

then conclusion is

$$\lim_{\ell\to\infty} \lim_{\Lambda \uparrow \mathbb{Z}^d} \frac{1}{Z(\Lambda)} \int e^{-\frac{1}{2}(\nabla\phi, \nabla\phi)} F^\Lambda e^{-\langle \phi, h_\ell \rangle} \qquad \text{(scaling)}$$

$$= e^{-\frac{1}{2}\iint h(x) \frac{1}{\gamma |x-y|^{d-2}} h(y)\, dx\, dy}$$

$(*)$ chosen so conclusion holds for case $F_x = 1$. For $d=2$ $|x-y|^{-(d-2)}$ is replaced by $\ln |x-y|$ and I have left out some factors of 2π in RHS.

Idea of proof

For $\gamma > 0$, not yet determined, write LHS of (scaling) as

$$Z(\Lambda)^{-1} \int e^{-\gamma/2 (\nabla\phi, \nabla\phi)} \, e^{-\langle \phi, h_\ell \rangle} \, F_\gamma^{\Lambda}$$

$\uparrow$ $\exp\left(-\frac{(1-\gamma)}{2}(\nabla\phi, \nabla\phi)\right)$ into $\nearrow$

Get rid of $\langle \phi, h_\ell \rangle$ by completing square

$$\phi_x = \tilde{\phi}_x + \underbrace{\sum_y (-\gamma\Delta)^{-1}_{xy} \, h_\ell(y)}_{uh(y),} \qquad u = (-\gamma\Delta)^{-1}$$

to obtain

$$\Big(\underbrace{Z(\Lambda)^{-1} \int e^{-\gamma/2 (\nabla\phi, \nabla\phi)}}_{\mathbb{E}} \; \underbrace{F_\gamma(\cdot + uh_\ell)^{\Lambda}}_{(\tau F)^{\Lambda}} \Big) \; \underbrace{e^{\frac{1}{2}(h_\ell, u h_\ell)}}_{\to \text{ RHS (scaling) as } \ell \to \infty}$$

with $\tau = \tau_{uh_\ell}$ = translate ϕ by uh_ℓ.

Therefore it suffices to prove $\exists \gamma$ s.t.

$$\lim_{\ell \to \infty} \lim_{\Lambda \nearrow} \frac{\mathbb{E}\left(\tau_{uh_\ell} F_\gamma\right)^{\Lambda}}{\mathbb{E} \, F_\gamma^{\Lambda}} = 1 \qquad \text{(target)}$$

Instead will take easier limit $\ell = L^N$, $\Lambda \in \mathcal{B}_N$ where diam $(\Lambda) = L^N$, $\ell = L^N$, $N \to \infty$.

The point of this simplification is that we are going to evaluate $\mathbb{E}$ as a sequence of conditional expectations

$$\begin{array}{ccccccccc} \phi & = & \zeta_1 & + \zeta_2 & + \cdots + & \zeta_{N-1} & + & \phi_N & = \zeta_N + \zeta_{N+1} + \cdots \\ & & \mathbb{E}_1 & \mathbb{E}_2 & & \mathbb{E}_{N-1} & & \mathbb{E}_N & \end{array}$$

and we can write

$$\phi + u h_{\ell} = \zeta_1 + \zeta_2 + \cdots + \zeta_{N-1} + (\phi_N + u h_{L^N})$$

because $u h_{L^N}$ regarded as a test function,

$$g_x = u h_{L^N}(x_1) \, u h_{L^N}(x_2) \cdots u h_{L^N}(x_p),$$

has norm $\|g\|_{\mathfrak{B}} = O(1)^{\dagger}$ as $N \to \infty$. Thus,

$$\mathbb{E}_N \mathbb{E}_{N-1} \cdots \mathbb{E}_1 \, \tau I \circ \tau K(\Lambda) = \tau I_N \circ \tau K_N(\Lambda)$$

$$= \tau I_N(\Lambda) + \tau K_N(\Lambda) \qquad \Lambda \in \mathfrak{B}_N !$$

$$= I_N(\Lambda, u h_{L^N}) + K_N(\Lambda, u h_{L^N})$$

$$= \underbrace{I_N(\Lambda, 0) + K_N(\Lambda, 0)}_{\text{same as denominator in (target)}} + \text{Taylor expansion in powers of } u h_{L^N}$$

so it suffices to prove that $I_N \to 1$, $K_N \to 0$ in T_0 norm.

$\dagger$ see page 12

Choose

$$I_j(B) = e^{-\sum_{x \in B} \left(a_j (\nabla\phi)^2_x + b_j\right)}$$

Remark

1 has dimension L^0

$(\nabla\phi)^2$ dimension $L^{-2[\phi]-2} = L^{-2\frac{d-2}{2}-2} = L^{-d}$

By Definition at end of Lecture 2, 1 is relevent, $(\nabla\phi)^2$ is marginal, all other monomials which are even and Euclidean invariant, such as

$$(\nabla\phi)^4 \qquad L^{-4\left(\frac{d-2}{2}+1\right)} = L^{-2d}$$

are irrelevent. End of remark.

$$\text{Write} \qquad F_{\gamma x} = I_{\underset{j=0 \text{ block}}{0}}(\{x\}) + \left(F_{\gamma x} - I(\{x\})\right)$$

Choose a_0 so $F_{\gamma x} - I\{x\} = O(\nabla\phi)^4$

Binomial Thm: $F_\gamma^\Lambda = I_0^\circ K_0(\Lambda)$

$F \simeq 1$ hypothesis $\Rightarrow$ ϵ_K small, start inductive use of Theorems 1,2,3.

Coupling constants a_j, b_j flow. Choose a_0, b_0 s.t $(a_j, b_j) \to (0,0)$ (See Park City Notes). By our general formalism of lectures 3,4 $\|K_j\|_j \to 0$ so we have $I_N \to 1$, $K_N \to 0$ in T_0 norm

To check the norm $\|g\|_\Phi$ of

$$g_x = uh_{L^N}(x_1) \cdots uh_{L^N}(x_p)$$

Since $\|g\|_\Phi = \|uh_{L^N}\|_\Phi^p$ regarding uh_{L^N} as a test function concentrated on sequences of length $p=1$ it suffices to check $\|uh_{L^N}\|_\Phi$. The $L^{jN}\nabla$ parts of the Φ norm commute through u and the scaling in h_{L^N} makes them bounded by derivatives of $h \in C_0^\infty$. So we consider $\alpha = 0$ in $\| \ \|_\Phi$, leading to

$$\left| L^{N[\phi]} u h_{L^N}(x) \right|$$

$$= L^{N[\phi]} L^{-\frac{d+2}{2}N} \left| \sum_{y\in\Lambda} u(x,y)\, h\left(\frac{y}{L^N}\right) \right|$$

$$\leq L^{N[\phi]} L^{-\frac{d+2}{2}N} \left(\sum_{y\in\Lambda} O(|x-y|^{-(d-2)}) \right) \operatorname{Sup}|h(y)|$$

$$= L^{N[\phi]} L^{-\frac{d+2}{2}N} L^{2N} \operatorname{Sup}|h| \qquad (\operatorname{diam}(\Lambda) = L^N)$$

Put in $[\phi] = \frac{d-2}{2}$.

$$= \operatorname{Sup}|h|. \quad \checkmark$$