Lecture 4

In the last lecture we introduced "coordinates" (I, K) where

$$I = (I(B),\ B \in \mathcal{B})$$
$$K = (K(X),\ X \in \mathcal{P})$$

$I(B)$ is, in applications, a known function of the fields $(\Phi_x,\ x \in B^*)$. For example,

$$I(B) = e^{-\sum_x (g\,\Phi_x^4 + a\Phi_x^2 + b)} \qquad (I\text{-choice})$$

while $K(X)$ is $(\Phi_x,\ x \in X^*)$ measurable. It is an error term subject to an estimate of the form

$$\| K(X) \| \leq \epsilon_K^{n_X} A^{-|X|}. \qquad |X| = \text{Card } \mathcal{B}(X)$$

is # of blocks in X

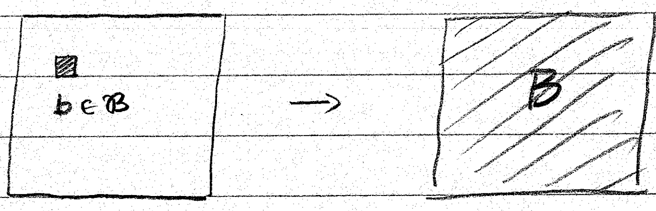
(I, K) determine an admissible <u>interaction</u>

$$I \circ K(\Lambda) = \sum_{X \in \mathcal{P}(\Lambda)} K(X)\, I^{\Lambda \setminus X}$$

More than one pair (I, K) can represent the same interaction. Strange to say, this is a GOOD THING because, as we will see today, we can exploit it to trade in an (I, K) we don't like because K is large for a better pair where K is smaller.

First some general remarks about the last lecture in order to motivate the need for the main result of todays lecture.

We defined $X \to \bar{X}$ on sets $X \in \mathcal{P}$ by $\bar{X}$ = smallest $U \in \mathcal{P}'$ s.t. $U \supset X$. This map is many to one, e.g.



All $b \in \mathcal{B}(B)$ map to B, so B is $\bar{b}$ for L^d different $b \in \mathcal{B}$. I will refer to this $O(L^d)$ preimages effect as the "entropy of coarse graining". We have to keep it in mind in the context of a special class of sets, the <u>small sets</u>, defined by

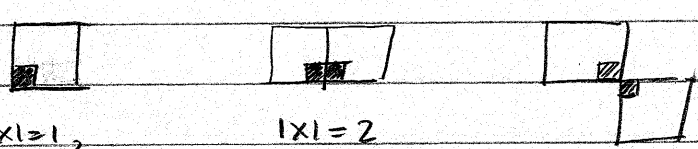
$$\mathcal{S} = \{ X \in \mathcal{P}, \ X \text{ connected}, \ |X| \le 2^d \}$$

There is a completely analogous $\mathcal{S}'$ which are the small sets on the next scale, $\{X \in \mathcal{P}', \ X \text{ connected}, \ |X|' \le 2^d\}$ where for $X \in \mathcal{P}'$,

$$|X|' = \text{Card } \mathcal{B}'(X)$$

is the number of blocks with side L^{j+1} in X.

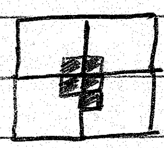
The small sets have the dangerous property that their closures may not look smaller



$|X|=1$, $|\bar{X}|'=1$ $\qquad$ $|X|=2$, $|\bar{X}|'=2$ $\qquad$ $|X|=2$, $|\bar{X}|'=2$

whereas, by Lemma Geometric of last lecture, if $X \notin S$, then

$$|\bar{X}| < |X|$$



$|X| = 5$

$|\bar{X}| = 4$

The entropy of coarse graining is dangerous for constructions such as

$$\bar{K}(U) = \sum_{\substack{X \in S \\ \bar{X} = U}} K(X)$$

when we substitute in our typical bound

$$\|K(X)\| \le \epsilon_K A^{-|X|}$$

because the triangle inequality says

$$\|\bar{K}(U)\| \le \epsilon_K \sum_{\substack{X \in S \\ \bar{X} = U}} A^{-|X|}$$

$$\le \epsilon_K \sum_{\substack{X \in S \\ \bar{X} = U}} A^{-|\bar{X}|'} = \epsilon_K A^{-|U|'} \underline{\underline{O(L^d)}}$$

so $\bar{K}$ has $\epsilon_{\bar{K}} = O(L^d)\, \epsilon_K$ whereas we want <u>contractive</u> estimates.

On the other hand if we consider the part of K that does not live on S, in the same type of operation

$$\bar{K}(U) = \sum_{\substack{X \text{ connected} \\ X \notin S \\ \bar{X} = U}} K(X)$$

then the fact that $|\bar{X}|' \leq |X|$ helps offset the entropy of coarse graining by releasing a helpful factor of A,

$$\| \bar{K}(U) \| \leq \sum_{\substack{X \text{ connected} \\ X \notin S \\ \bar{X} = U}} \epsilon_K \, A^{-|X|}$$

$$\leq \cdots\cdots \epsilon_K \, A^{-|\bar{X}|'} \, A^{-1}$$

$$= \epsilon_K \, O(L^d) \, A^{-1} \, A^{-|U|'}$$

and so we take A large so that $O(L^d) A << 1$ and find that $\epsilon_{\bar{K}} << \epsilon_K$ and therefore have contraction. If you now look back at Theorem 2 you will see that this was the principle we used. It means that the map of Theorem 1

$$(I, K) \to (I', \tilde{K}) \quad \text{s.t.} \quad \mathbb{E}\, K \circ I(\Lambda) = \tilde{K} \circ I'(\Lambda)$$

contracts the part of K that does not live on $X \in S$ and all our difficulties with entropy of coarse graining are in $\tilde{I}$. We will not be able to prove that $\tilde{I}$ obeys a better estimate than K without a good new idea and this idea is to change $(I', \tilde{K})$ to a better (I', K') which represents the same interaction.

We set everything up on the $\mathcal{P}'$ scale because we want to use the result on $(I', \tilde{K})$ but there is nothing specific to $(I', \tilde{K})$ in the following construction. It is a general result about the IoK representation.

Let

$$\hat{\mathcal{L}} : \mathcal{P}' \to \{\text{measurable fns}\}, \qquad \hat{\mathcal{L}} \text{ factors over } \mathcal{P}'$$

$$\hat{\mathcal{L}}(X) = 0 \quad \text{if } X \text{ connected}, \; X \notin S'$$

$$\sum_{\substack{X \in S' \\ X \supset B}} \frac{1}{|X|'} I'^{-X} \hat{\mathcal{L}}(X) = 0, \qquad B \in \mathcal{B}' \qquad (\Sigma = 0)$$

Estimates rely on the following standard structure familiar from last lecture: define $\epsilon_{\hat{\mathcal{L}}}$, ϵ_R, α best constants s.t.

$$\|\hat{\mathcal{L}}(X)\| \le \epsilon_{\hat{\mathcal{L}}}^{n_X} A^{-|X|'}, \qquad X \in \mathcal{P}'$$

$$\|\tilde{K}(X) - \hat{\mathcal{L}}(X)\| \le \epsilon_R^{n_X} A^{-|X|'}, \qquad \text{"}$$

$$\alpha \ge \|I'(B)\| \qquad B \in \mathcal{B}'$$

Theorem 3

For $A \geq 3$ let

$$\beta(A) = c(S) \, (\alpha A)^{2^d c(S)}$$

where $c(S) = \text{Card}\{X \in S, X \supset B\}$. If

$$\epsilon_R + \beta(A) \, \epsilon_{\hat{\mathcal{L}}} \leq 1$$

then $\exists \; K'$ such that

1) $K' \circ I'(\Lambda) = \tilde{K} \circ I'(\Lambda)$

2) $$\| K'(X) - (\tilde{K}(X) - \hat{\mathcal{L}}(X)) \| \leq \left(\epsilon_R + \beta(A) \, \epsilon_{\hat{\mathcal{L}}}\right)^2 (A/3)^{-|X|'}$$

This says that $K' \approx \tilde{K} - \hat{\mathcal{L}}$ because ϵ_R, $\epsilon_{\hat{\mathcal{L}}}$ will be small, so $(\epsilon_R + \beta(A) \, \epsilon_{\hat{\mathcal{L}}})^2$ is even smaller.

The loss $A \to A/3$ is not serious because we gained $A \to A^{1+\delta}$ in Theorem 2 and for A large this is a much larger gain for the composition

$$(I, K) \to (I', \tilde{K}) \longrightarrow (I', K')$$

According to Theorem 1, $\forall (I,K)$ on $\mathcal{B}\times\mathcal{P}$,

$$\mathbb{E}_+ K\circ I(\Lambda) = \tilde{K}\circ I'(\Lambda)$$

and Theorem 2 allows us to replace the complicated formula for $\tilde{K}$ by the leading terms

$$\tilde{\mathcal{L}}(U) = \sum_{X\in\bar{S}(U)} \mathbb{E}_+ K(X)\ \tilde{I}^{U\setminus X}$$

$$+ \sum_{\substack{X\in\bar{\mathcal{P}}(U)\\ |X|=1,2}} \mathbb{E}_+ \delta I^X\ \tilde{I}^{U\setminus X}$$

where

$$\bar{S}(U) = \{X\in S,\ \bar{X}=U\}$$

$$\delta I = I-\tilde{I},\quad \tilde{I}\ \ \Phi'\text{ measurable, arbitrary.}$$

<u>The Idea</u>

Choose $\tilde{I}$ so that $(\Sigma=0)$ holds for $\tilde{\mathcal{L}}$, namely,

$$\sum_{\substack{U\in S'\\ U\supset B}} \frac{1}{|U|'}\ I'^{-U}\ \tilde{\mathcal{L}}(U) = 0,\qquad B\in\mathcal{B}' \qquad (\tilde{\mathcal{L}}\text{-condition})$$

Then we can choose $\hat{\mathcal{L}}=\tilde{\mathcal{L}}$. Then the map $(I,K)\to(I',\tilde{K})\to(I',K')$ is contractive on K because K' only contains the tiny error terms of Theorems 2, 3.

After chasing through the definitions you will find that ($\tilde{I}$-condition) says

$$\sum_{b \in \mathcal{B}(B)} \tilde{I}(b)^{-1} \, \mathbb{E}_+ \left(I(b) - \tilde{I}(b) + K(b) \right) \quad +$$

$$\sum_{\substack{X \in \mathcal{P} \\ \bar{X} \ni B}} \frac{1}{|X|'} \, \tilde{I}^{-X} \left(1_{\substack{X \in S \\ |X| \geq 2}} \mathbb{E}_+ K(X) + 1_{\substack{|X| = 1,2 \\ \bar{X} \in S'}} \mathbb{E}_+ \delta I^X \right)$$

$$= 0, \qquad B \in \mathcal{B}' \qquad (\tilde{I}\text{-formula})$$

This is an implicit equation for $\tilde{I}$. If we ignore the (smaller) second line it says, keeping in mind that $\tilde{I}$ is Φ' measurable,

$$\tilde{I}(b) = \mathbb{E}_+ \left(I(b) + K(b) \right) \qquad (1^{st} \text{order } \tilde{I}\text{-formula})$$

The $\mathbb{E}_+ \delta I^X$ in the second line, $|X| = 2$, brings in 2^{nd} order perturbation theory as we derived in lecture 2. The $\mathbb{E}_+ K(X)$, captures what was formerly known as $O(g^3)$ in perturbation theory, Lecture 2.

If we want $\tilde{I}$ to have the same form as we have assumed for I, c.f. (I-choice) for example, then we will not be able to solve ($\tilde{I}$-formula) exactly, because, for example $E_+ I(b)$ according to perturbation theory in Lecture 2 has Φ'^6 terms which were not in (I-choice).

The advantage of $\tilde{I}$ keeping the same form as I is that then I' will have the same form as I because $I'(B) = \prod_{b \in \mathcal{B}(B)} \tilde{I}(b)$.

If we do not exactly solve ($\tilde{I}$-formula), we can only cancel some part $\hat{\mathcal{L}} \neq \tilde{\mathcal{L}}$ of $\tilde{\mathcal{L}}$. This will make it harder to prove that the composite map $(I, K) \longrightarrow (I', \tilde{K}) \longrightarrow (I', K')$ is contractive on the K component, but at least the formalism provides a systematic framework to pose and answer this question.

An affirmative answer means that (I-choice) is rich enough to capture the scaling limit. Wilson, with his definition of relevant and marginal monomials (end of Lecture 2), gave us the conjectural prescription for building the correct I. so Theorems 2, 3 are building blocks for proving he is correct. At the moment have only proved that he is correct for special cases. My Park City notes discuss one (anharmonic lattice $\equiv$ dipole gas) and I may have time in the next lecture to talk about that.

The rest of this lecture is devoted to parts of the proof of Theorem 3.

The key is first to understand

Special case $\tilde{K} = \hat{\mathcal{L}}$. $(\epsilon_R = 0)$. Theorem 3 claims that

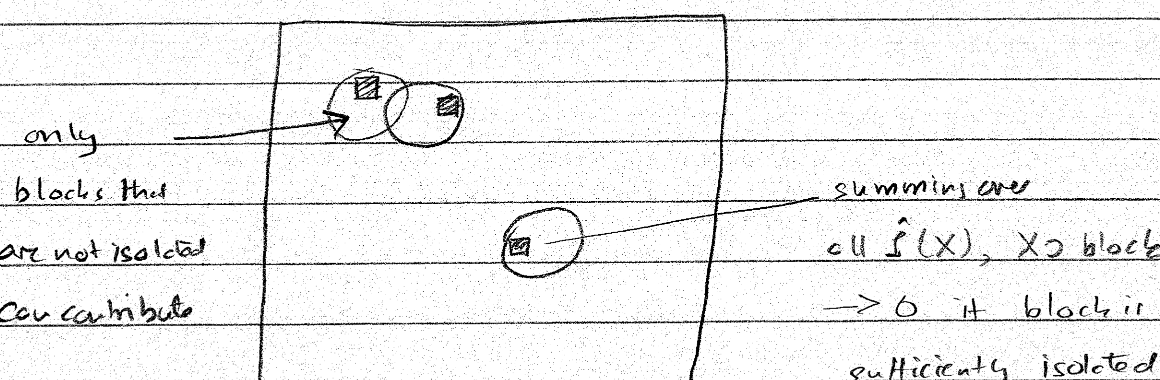
$$I' \circ \tilde{K}(\Lambda) = I' \circ K'(\Lambda)$$

with

$$\| K'(X) \| \leq \left(\beta(A)\, \epsilon_{\hat{\mathcal{L}}}\right)^2 (A/3)^{-|X|'}$$

Key point: why $O(\epsilon_{\hat{\mathcal{L}}}^2)$ and not $O(\epsilon_{\hat{\mathcal{L}}})$ { I will derive a formula for K' and use it to answer this.

We need the picture that "says it all" (a poor imitation of the style of David Williams)

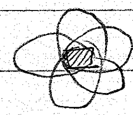


but just in case this picture did not say it all, here follow some details.

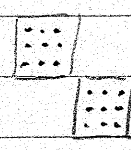
Small set neighbourhood

For $B \in \mathcal{B}'$ define the small set neighbourhood,

$$B^* = \bigcup \{ Y \in S', Y \supset B \} .$$



If $Y_1, Y_2, \cdots, Y_n$ are connected sets in $\mathcal{P}'$ and $Y_i \cup Y_j$ is disconnected for every $i \neq j$ we say $Y_1, \cdots, Y_n$ do not touch



these blocks touch.

Pinned components For $n = 1, 2, \cdots$, say that

$$\{ (Y_1, B_1), (Y_2, B_2), \cdots, (Y_n, B_n) \} \in PC$$

if $Y_1, Y_2, \cdots, Y_n \in S'$, do not touch, and $B_i \in \mathcal{B}'(Y_i)$ for $i = 1, \cdots, n$.

Think of breaking a set $Y \in \mathcal{P}'$ into its connected components and for each connected component, selecting a block in the component. In the case that all components are small sets this construction generates an element $y \in PC$.

Lemma PC. If $\hat{\mathcal{L}}(X) = 0$ when X is connected, not in S_3' then

$$\sum_{X \in \mathcal{P}} \hat{\mathcal{L}}(X)\, I^{\Lambda \setminus X}$$

$$= \sum_{\mathcal{Y} \in PC} \left(\prod_{(Y,B) \in \mathcal{Y}} \frac{1}{|Y|'} \hat{\mathcal{L}}(Y) \right) I^{\Lambda \setminus U_{\mathcal{Y}}}$$

where

$$U_{\mathcal{Y}} = \bigcup \{ Y : (Y,B) \in \mathcal{Y} \}$$

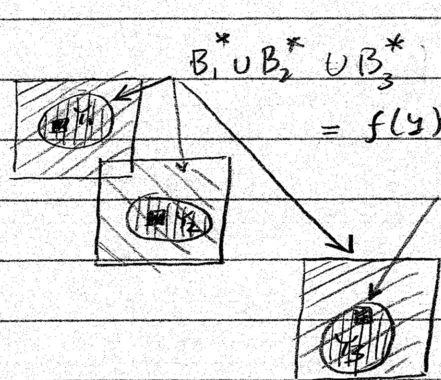
Proof $\sum_{\mathcal{Y}}$ includes summing over $B \in \mathcal{B}'(Y)$ which is cancelled by $\frac{1}{|Y|'}$ so RHS is summing over $X \in \mathcal{P}'$ by thinking of it as a sum over the components of X.

∎

Let $f : PC \to \mathcal{P}'$ be defined by

$$f(\mathcal{Y}) = \bigcup \{ B^* : (Y,B) \in \mathcal{Y} \}$$

The idea is to glue into one set whenever two B's have overlapping B^*'s



$B_1^* \cup B_2^* \cup B_3^*$ is a coarse graining of $\{Y_1, Y_2, Y_3\}$ with a nice property: all the possible choices of Y_3, for B_3 fixed can be easily resummed, because they cannot "bump into" Y_1 or Y_2 since $Y_3 \subset B_3^*$ and B_3^* is disjoint from B_2^* and B_1^*, which imprison Y_2 and Y_1.

<u>Lemma $\hat{\mathcal{L}}$</u> Given $\hat{\mathcal{L}}$ as in Lemma PC, for $W \in \mathcal{P}'$, let

$$K'(W) = \sum_{\mathcal{Y} \in f^{-1}(W)} \left(\prod_{(Y,B)\in \mathcal{Y}} \tfrac{1}{|Y|}\,' \hat{\mathcal{L}}(Y) \right) I^{-W \setminus U_{\mathcal{Y}}}$$

(empty sum $= 0$), then

$$I' \circ \hat{\mathcal{L}}(\Lambda) = I' \circ K'(\Lambda)$$

and K' factors.

<u>Proof</u> Write $\sum_{\mathcal{Y}\in PC} = \sum_{W\in\mathcal{P}} \sum_{\mathcal{Y}\in f^{-1}(W)}$ in Lemma PC and $I^{\Lambda \setminus U_{\mathcal{Y}}} = I^{\Lambda\setminus W}\, I^{W \setminus U_{\mathcal{Y}}}$.

Factorisation is for you to prove, but notice that $f^{-1}(W_1 \cup W_2)$ is easily expressed in terms of $f^{-1}(W_1)$ and $f^{-1}(W_2)$ if W_1 does not touch W_2 ▨

<u>Lemma</u> If

$$\sum_{\substack{Y \in S\\ Y \ni B}} \tfrac{1}{|Y|} I^{-Y} \hat{\mathcal{L}}(Y) = 0$$

then the contribution to $K'(W)$ from $\mathcal{Y}$s with Card $\mathcal{Y} = 1$ is zero.

<u>Proof</u> Card $\mathcal{Y} = 1$ means $\mathcal{Y} = \{(Y,B)\}$, and $\mathcal{Y} \in f^{-1}(W)$ means $B^* = W$ so the contribution of $\mathcal{Y} \in f^{-1}(W)$ with Card$\mathcal{Y} = 1$ is

$$\sum_{(Y,B)} 1_{B^*=W} \tfrac{1}{|Y|}\,\hat{\mathcal{L}}(Y)\, I^{W\setminus Y}$$

$$= I^{W} \sum_{B:\, B^*=W} \sum_{Y \ni B} \tfrac{1}{|Y|}\, I^{-Y} \hat{\mathcal{L}}(Y) = 0$$

▨

This last Lemma explains why K' is $O(\epsilon_{\hat{\mathcal{L}}}^2)$ and here is the

<u>Proof</u> By formula for K', previous lemma, hypotheses,

$$\|K'(W)\| \leq \sum_{\substack{y \in f^{-1}(W) \\ \text{Card } y > 1}} \left(\prod_{(Y,B) \in y} \underbrace{\frac{1}{|Y|'}}_{\leq 1} \epsilon_{\hat{\mathcal{L}}} A^{-|Y|'} \right) \alpha^{|W \setminus \cup_y|'}$$

$$\leq A^{-|W|'} \sum_{\substack{y \in f^{-1}(W) \\ \text{Card } y > 1}} \epsilon_{\hat{\mathcal{L}}}^{\text{Card } y} (\alpha A)^{|W|'}$$

$$\leq A^{-|W|'} \sum_{\cdots} \left(\epsilon_{\hat{\mathcal{L}}} (\alpha A)^{2^d c(S)} \right)^{\text{card } y}$$

In the last step we used $W = \cup\{B^* : (Y,B) \in y\}$ so $|W|' \leq \text{Card } y \; |B^*|'$ $= (\text{Card } y) \; \text{Card}\{X \in S, X \supset B\} \; \underbrace{2^d}_{}$ — size of small set.

The sum over $y \in f^{-1}(W)$, $\text{Card } y > 1$: Write this as a sum over a set $X \subset W$ such that $W = \cup\{B^* : B \in \mathcal{B}(X)\}$, and for each $B \in \mathcal{B}(X)$, a sum over $Y \supset B$, $Y \in S$. Each of these $Y \supset B$ sums has at most $c(S)$ summands (definition of $c(S)$) and there are $|X|'$ Y sums.

$$\leq A^{-|W|'} \sum_{\substack{X \subset W \\ |X| > 1}} \Big(\underbrace{\epsilon_{\hat{\mathcal{L}}} (\alpha A)^{2^d c(S)} \; c(S)}_{= \beta(A) \epsilon_{\hat{\mathcal{L}}}} \Big)^{|X|'}$$

$$\leq A^{-|W|'} \left(\beta(A) \epsilon_{\hat{\mathcal{L}}}\right)^2 \sum_{\substack{X \subset W \\ \cancel{|X| > 1}}} 1 \qquad (*)$$

$$= A^{-|W|'} \left(\beta(A) \epsilon_{\hat{\mathcal{L}}}\right)^2 \; 2^{|W|'}$$

$$= (A/2)^{-|W|'} \left(\beta(A) \epsilon_{\hat{\mathcal{L}}}\right)^2 \qquad ▨$$

The reason why this proof of the special case $(\tilde{K} = \hat{\mathcal{L}})$ ended up with a bound by

$$(A/2)^{-|W|'}$$

whereas the Theorem 3 only claimed the weaker $(A/3)^{-|W|'}$ is that when we go through the above argument but putting in the contributions from $\tilde{K} - \hat{\mathcal{L}}$, we find at stage * that there is a sum

$$\sum_{\substack{X, U_R \text{ disjoint} \\ \subset W}} 1 \leq 3^{|W|'}$$

where U_R is the set occupied by $\tilde{K} - \hat{\mathcal{L}}$ and so now we have to use the trinomial theorem. Try, as an exercise, to put in $\tilde{K} - \hat{\mathcal{L}}$.