Lecture 3

For the hierarchical lattice we have defined a map

$$I \longmapsto I' \qquad \text{where} \quad I = (I_x, x \in \Lambda)$$
$$I' = (I'_x, x \in L^{-1}\Lambda)$$

by

$$E(I^{\Lambda} \mid \Phi') = I'^{\,L^{-1}\Lambda}$$

($\Phi = \Phi' + \zeta$, integrate out ζ)

We have seen how to calculate, within perturbation theory, I' when $I_x = e^{-V_x}$ and V is a polynomial such as

$$V_x = g\,\Phi_x^4 + a\,\Phi_x^2 + b \qquad (\text{quartic})^{\dagger}$$

and when we include in $I \to I'$

$$\Phi'_x \overset{\mathcal{D}}{=} L^{-[\phi]}\,\Phi_{L^{-1}x} \qquad \text{scaling}$$

perturbation theory predicts that $I \to I'$ is "mostly" a flow in the "coupling constants" $(g, a, b) \longrightarrow (g', a', b')$; additional terms such as Φ'^6 appear in $I'_x = e^{-V'_x}$ but this comes with prefactor $L^{d-6[\phi]}$ which keeps it small, for example, when $d = 4$, $[\phi] = 1$.

$\dagger$ Not using Wick powers is equivalent to a linear transformation on (g, a, b).

To formulate a theorem we have to control errors. The error will be called K. We make a shift in notation:

$$I_x = e^{-V_x}$$

V_x polynomial
finite # of coefficients
e.g. (g, a, b)

RG is now a map

$$RG: (I, K) \longrightarrow (I', K')$$

$K = (K_x; x \in \Lambda)$
K_x is Φ_x measurable

such that

$$\mathbb{E}\left((I+K)^{\Lambda} \mid \Phi' \right) = (I'+K')^{L^{-1}\Lambda}$$

for hierarchical models, and, as we shall see, a natural generalisation extends this equation to Euclidean models.

The objective is to construct a natural (class of) norms on measurable functions, in particular K_x, and domain

$$\mathcal{D} = \{ (\overbrace{g, a, b}^{I}, K) : \ g > 0, \ |a|, \ |b|, \ \|K\| \text{ small} \}$$

s.t. $\quad RG: \mathcal{D} \to \mathcal{D}$ (2)

Today we will prove two theorems which get us halfway and next lecture will be the remaining half.

For hierarchical lattice our map is $(I, K) \to (I', K')$ such that

$$E\left((I+K)^{\Lambda} \mid \Phi'\right) = (I'+K')^{L^{-1}\Lambda}$$

How can we generalise this to Euclidean lattices?

<u>Binomial Theorem</u>

$$(I+K)^{\Lambda} = \sum_{X\subset\Lambda} K^X I^{\Lambda\setminus X}$$

$$\left(K^X = \prod_{x\in X} K_x, \qquad I^X = \prod_{x\in X} I_x\right)$$

On the hierarchical lattice every point x is a connected component $\{x\}$ of $X \subset \Lambda$, in the sense that for x, y distinct points in X, $|x-y| \geq L$, so we can interpret K^X as a function $X \to K(X)$ with the property

$$K(X) = \prod_{Y\ \text{c.c.}\ X} K(Y) \qquad \left(K \text{ Factorises}\right)$$

c.c. "connected component of"

Defn $Z(\Lambda)$ is an admissible interaction if for some (I, K)

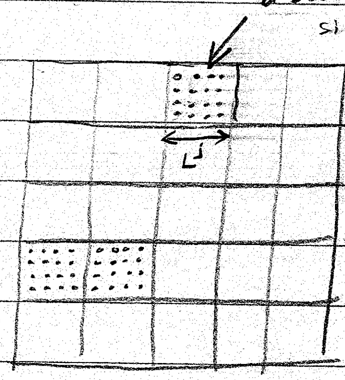
$$Z(\Lambda) = \sum_{X \in \mathcal{P}(\Lambda)} K(X)\, I^{\Lambda \setminus X}$$

K factorises

$K(X)$ measurable w.r.t $(\Phi_x, x \in X^*)$

$(X^* = X$ for the moment$)$

Simple cubic lattice $\Lambda \subset \mathbb{Z}^d$



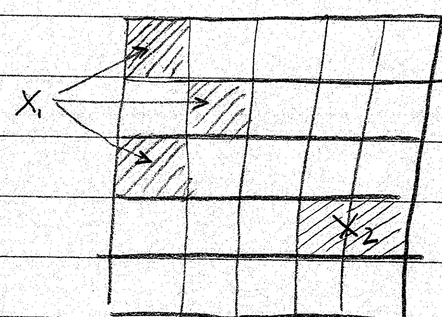
$X \in \mathcal{P}(\Lambda)$ if X is a union of blocks $B \in \mathcal{B}(\Lambda)$. Write $\mathcal{B} = \mathcal{B}(\Lambda)$.

Scale L^0: $\mathcal{B} = \{\{x\}, x \in \Lambda\}$

Scale L^j: $\mathcal{B} = \{B:$ disjoint "blocks" partitioning $\Lambda\}$

Factorisation (pictorial example)

$K(X) = K(X_1)\, K(X_2)$



$$I^{\Lambda \setminus X} = \prod_{B \in \mathcal{B}(\Lambda \setminus X)} I(B)$$

X has two connected components X_1, X_2

$I(B)$ measurable w.r.t $(\Phi_x, x \in B^*)$.

Notation $K \circ I\,(X) = \sum_{Y \in P(X)} K(Y)\, I^{X \setminus Y}$

Example (Binomial) Just to get used to notation:

Given $\tilde{I} = (\tilde{I}(B),\ B \in \mathcal{B})$

$SI = \left(SI(B),\ B \in \mathcal{B}\right)$

Define

$$\tilde{I}(X) = \tilde{I}^X, \quad SI(X) = SI^X$$

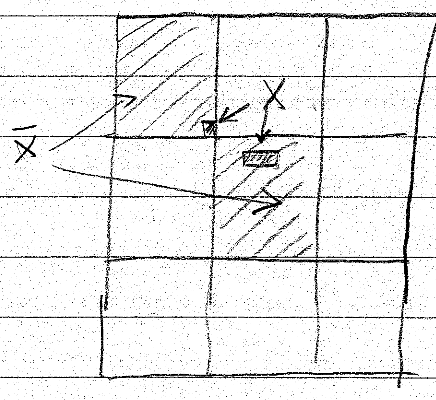
then

$$(\tilde{I} + SI)^X = \tilde{I} \circ SI\,(X) \qquad \text{(Binomial thm)}$$

Example (closure)

Given $X \in \mathcal{P}$ let $\bar{X} \in \mathcal{P}'$ be the smallest set in $\mathcal{P}'$ (unions of blocks of next scale) that contains X and write $X \in \bar{\mathcal{P}}(U)$ if $X \in \mathcal{P}$ and $\bar{X} = U$.

Given I, K, let



$$\bar{I}(B) = \prod_{b \in \mathcal{B}(B)} I(b), \quad B \in \mathcal{B}'$$

$$\bar{K}(U) = \sum_{X \in \bar{\mathcal{P}}(U)} K(X)\, I^{U \setminus X}$$

then

$$K \circ I\,(\Lambda) = \bar{K} \circ \bar{I}\,(\Lambda).$$

Proof:

$$K \circ I\,(\Lambda) = \sum_{X \in P} K(X)\, I^{\Lambda \setminus X}$$

$$= \sum_{U \in P'} \sum_{X \in \bar{P}(U)} K(X)\, \underbrace{I^{\Lambda \setminus X}}_{I^{\Lambda \setminus U}\; I^{U \setminus X}}$$

$$= \sum_{U \in P'} \underbrace{\left(\sum_{X \in \bar{P}(U)} K(X)\, I^{U \setminus X} \right)}_{\bar{K}(U)} \underbrace{I^{\Lambda \setminus U}}_{\bar{I}^{\Lambda \setminus U}}$$

$$= \bar{K} \circ \bar{I}\,(\Lambda)$$

∎

Furthermore (you do it!)

$\bar{K}$ factorises (on P')

Assume ζ is Gaussian with covariance $C = C_{j+1}$ s.t.

$$C(x,y) = 0 \quad \text{if } |x-y| \geq L^{j+1}/2 \qquad \text{(finite range)}$$

where L^j is the diameter of the blocks in $\mathcal{B}$. Let

$$\mathbb{E}_+(\circ) = \mathbb{E}(\circ \mid \Phi')$$

and let $\tilde{I}$, δI be s.t.

(i) $I = \tilde{I} + \delta I$

(ii) $\tilde{I}$ Φ' measurable

<u>Theorem 1</u>

$$\mathbb{E}_+\left(K \circ I(\Lambda)\right) = \tilde{K} \circ I'(\Lambda)$$

where, for $U \in \mathcal{P}'$, $B \in \mathcal{B}'$,

$$\tilde{K}(U) = \sum_{W \in \overline{\mathcal{P}}(U)} \mathbb{E}_+(K \circ \delta I)(W) \; \tilde{I}^{U \setminus W}$$

$$I'(B) = \prod_{b \in \mathcal{B}(B)} \tilde{I}(b)$$

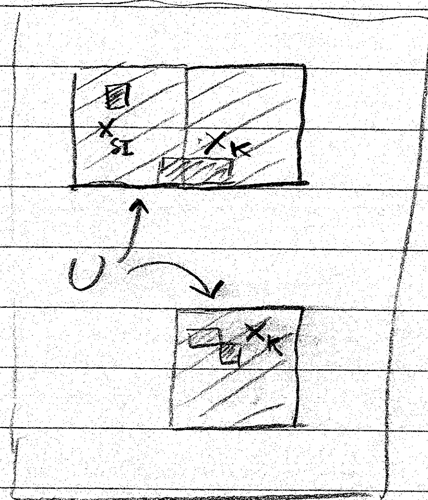
<u>Remark</u> In example closure $\overline{K}$ factorises on the next scale so $\mathbb{E}_+\overline{K}(U)$ factorises on the next scale because the connected components of $U \in \mathcal{P}'$ are separated by $\geq L^{j+1}$ $>$ range C.

Proof of Theorem 1

$$K \circ I(\Lambda) = K \circ (\tilde{I} + \delta I)(\Lambda)$$

$$= \underbrace{(K \circ \delta I)}_{J} \circ \tilde{I}(\Lambda) \qquad \text{(binomial)}$$

Check that J factorises. Then by example (closure)



$$= \bar{J} \circ I'(\Lambda)$$

because $I' = \bar{\tilde{I}}$

$\bar{J}$ factorises on the next scale
I' is $\bar{\tilde{I}}'$ measurable so $\mathbb{E}, \bar{J}$ factorises by Remark and

$$\mathbb{E}\, K \circ I(\Lambda)$$

$$= \mathbb{E}\, \bar{J} \circ I'(\Lambda)$$

$$= (\mathbb{E}\, \bar{J}) \circ I'(\Lambda) \qquad (I' \ \bar{\tilde{I}}' \text{ measurable})$$

$$= \tilde{K} \circ I'(\Lambda)$$

$\blacksquare$

Example $\overline{K}$

Suppose we have a norm s.t. $\forall F_1, F_2$

$$\|F_1(X)\, F_2(Y)\| \le \|F_1(X)\|\, \|F_2(Y)\|$$
if $X \cap Y = \emptyset$ (Norm factorisation)

and, given $A \geq 1$, let

$$\alpha \geq \sup_{b \in \mathcal{B}} \|I(b)\|$$ (I stable)

ϵ_K be best constant s.t $\|K(X)\| \le \epsilon_K^{n_X} A^{-|X|}$

where $n_X = \#$ c.c. X, $|X| = \text{Card } \mathcal{B}(X)$. Define

$\mathcal{L}(U) =$ terms in $\overline{K}(U)$ with $n_X = 1$ (Leading terms)

Proposition $\exists \delta > 0$, $A_0 = A_0(d, \alpha, L)$ and $A^* = A^*(A, d, \alpha)$ s.t. for all $A \geq A_0$, for all ϵ_K with $A^* \epsilon_K \le 1$,

$$\|\overline{K}(U) - \mathcal{L}(U)\| \le (A^* \epsilon_K)^2 A^{-(1+\delta)|U|'}$$

where

$$|U|' = \text{Card } \mathcal{B}'(U)$$

Proof

$$\|\bar{K}(U) - \mathcal{L}(U)\| \le \sum_{* \begin{cases} X \in \bar{\mathcal{P}}(U) \\ n_X \ge 2 \end{cases}} \|K(X)\| \, \|\mathcal{I}^{U \setminus X}\|$$

$$\le \sum_{*} \epsilon_K^{n_X} A^{-|X|} \alpha^{|U \setminus X|} \qquad \text{insert } \beta^{-|X|} \beta^{|X|}$$

$$\le \left(\sup_{(*)} \epsilon_K^{n_X} (\beta A)^{-|X|} \right) \left(\sum_{X \in \mathcal{P}} \beta^{|X|} \alpha^{|U \setminus X|} \right)$$

$$= \left(\sup_{(*)} \epsilon_K^{n_X} (\beta A)^{-|X|} \right) (\beta + \alpha)^{|U|}$$

Lemma (Geometric)† $\exists\, c > 1$ $\quad |X| \ge c|\bar{X}|' - c\, 2^{d+1} n_X$

$$\le \sup_{(*)} \left(\epsilon_K (\beta A)^{c 2^{d+1}} \right)^{n_X} \left(\frac{(\beta A)^c}{(\beta + \alpha)^{L^d}} \right)^{-|U|'}$$

Choose $\beta = \alpha$, (or optimise). Choose $\delta > 0$ s.t. $1 + \delta = \frac{1+c}{2}$. For A large $\quad \frac{\beta^c A^c}{(\beta + \alpha)^{L^d}} \ge A^{1+\delta}$ because $c > 1 + \delta$

set $\quad A^* = (\beta A)^{c 2^{d+1}}$

∎

† Proved in my Park City Lecture Notes.

End of example $\bar{K}$

Example $\bar{K}$ is the model for the following bound on $\tilde{K}$ defined in Theorem 1. Besides norm factorisation we need the norm to satisfy

$$\| \mathbb{E}_+ F(X) \| \leq \alpha^{|X|} \| F(X) \| \qquad (\| \mathbb{E}_+ \|)$$

As for $\bar{K}$, $\mathcal{L}$ will be the leading terms in $\tilde{K}$. Different models would need different choices but as an illustration let

$$\tilde{\mathcal{L}}(U) = \sum_{\substack{X \in \bar{\mathcal{P}}(U) \\ |X| \leq 2^d \\ \text{connected}}} \mathbb{E}_+ \big(K(X) \big) \, \tilde{I}^{U \setminus X} \qquad \left(\text{Leading terms} \right)$$

$$+ \sum_{\substack{X \in \bar{\mathcal{P}}(U) \\ |X| = 1, 2}} \mathbb{E}_+ \big(\delta I^X \big) \, \tilde{I}^{U \setminus X}$$

Theorem 2 δ, A, A^* as in Example $\bar{K}$,

$$\| \tilde{K}(U) - \tilde{\mathcal{L}}(U) \| \leq \left(\epsilon_K \vee (A^* \epsilon_{\delta I})^3 \right) A^{-(1+\delta)|U|'}$$

where $\epsilon_{\delta I}$ is the best constant s.t. $\| \delta I(B) \| \leq \epsilon_{\delta I} A^{-1}$ for all $B \in \mathcal{B}$.

Proof very close to example $\bar{K}$. You could try it as an exercise; why does Lemma Geometric help out when X is connected, $|X| > 2^d$ and A is large?

When A is large the right hand side is a much stronger estimate than the ones we assumed for K so the point of Theorem 2 is that it shows a big contraction in $\tilde{K}-\tilde{\mathcal{L}}$. If we want to prove that $K \to \tilde{K}$ is a contraction then we only have to analyse the much simpler $\tilde{\mathcal{L}}$. In the next lecture we will be doing this.