Lecture II

Recall from lecture I,

$$\int d\mu_C(\phi)\ P = e^{\frac{1}{2}\Delta_C} P\Big|_{\phi=0}$$

$$\Delta_C = \frac{1}{2} \sum_{i,j\in\Lambda} C_{ij} \frac{\partial}{\partial\phi_i}\frac{\partial}{\partial\phi_j}$$

$$e^{\Delta_C/2} = \sum_{j\geq 0} \frac{1}{j!}\left(\frac{\Delta_C}{2}\right)^j$$

Reminder: P is polynomial

$$:P:_C = e^{-\frac{1}{2}\Delta_C} P, \qquad \frac{\partial}{\partial\phi} :P:_C = :\frac{\partial P}{\partial\phi}:_C$$

Examples

$$\int d\mu_C :P:_C = P\Big|_{\phi=0}$$

$$\int d\mu_C :P:_{U+C} = :P:_U\Big|_{\phi=0}$$

where U_{ij} is another positive-definite $\Lambda\times\Lambda$ matrix.

Today will be about the renormalisation group as the theoretical physicists understand it, so there will be lots of series, which are not convergent, but are asymptotic. I am not going to give proofs, but only examples of the use and variety of these series. The above examples of normal ordering play a significant role.

2.1 Feynman expansion

Let

$$\mathrm{Int} \overset{\text{def}}{=} \int d\mu_C(\zeta)\, e^{-g P(\zeta)}$$

when P is a polynomial which is bounded below and $g > 0$.

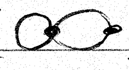
$$\overset{\text{DCT}}{=} \sum_{j=0}^{n} \int d\mu_C(\zeta) \frac{(-gP)^j}{j!} + O(g^{n+1})$$

notation for line above (DCT = dominated convergence)

$$\sim \sum_{j \geq 0} \int d\mu_C(\zeta) \frac{(-gP)^j}{j!}.$$

$$= \sum_{j \geq 0} \sum_{G \text{ on } j \text{ vertices}} I(G) \quad \text{by Wick Theorem}$$

This is a sum over all unlabelled graphs with weight including $\left(\tfrac{1}{2}\right)^{\#\text{ tadpoles}}$ $\frac{1}{|\mathrm{Aut}|}$ as before:

Example $I($$) = \left(\tfrac{1}{2}\right)^1 \left(\tfrac{1}{2}\right) \sum_{\text{indices}} C_{ij} \left.\frac{\partial^4(-gP)}{\partial\zeta_i\,\partial\zeta_j\,\partial\zeta_k\,\partial\zeta_l}\right|_0 C_{km}\, C_{ln} \left.\frac{\partial^2(-gP)}{\partial\zeta_m\,\partial\zeta_n}\right|_0$

!

Theorem $\ln I(g) \sim \sum_{G \text{ connected}} I(G)$ [Book. Manfred Salmhofer]

If P is replaced by $:P:_C$ then G has no tadpoles and at each vertex of degree r

$$\left.\frac{\partial^r P}{\partial\zeta\cdots\partial\zeta}\right|_{\zeta=0}$$

If $:P:_{r'+C}$ then no tadpoles and $\left. :\frac{\partial^r P}{\partial\zeta\cdots\partial\zeta}:_{r'}\right|_0$ at each vertex

2.2. The Laplace-Feynman Expansion (More general than Feynman exp.)

$$S: \mathbb{R}^\Lambda \to \mathbb{R}$$

is C^∞, has a unique non-degenerate minimum ϕ_{min}, $\lim_{\phi \to \infty} S(\phi) = +\infty$ and $\int e^{-S} d^\Lambda\phi < \infty$.

Given $\phi_0 \in \mathbb{R}^\Lambda$ let

$$A_{ij} = \left.\frac{\partial^2 S}{\partial\phi_i\,\partial\phi_j}\right|_{\phi_0},$$

$$N(\alpha) = \int e^{-\frac{1}{2}(\phi,\, \alpha A \phi)}\, d^\Lambda\phi$$

Then, with $\phi_0 = \phi_{min}$,

large deviations

CLT

$$\ln \int e^{-\alpha S} d^\Lambda \phi \sim -\alpha\, S(\phi_{min}) + \ln N(\alpha)$$

$$+ \underbrace{\sum_{G \text{ connected}} I(G)}_{(*)}$$

where

edge $\quad \alpha C$

vertex $\quad \left.\dfrac{\partial^r(-\alpha S)}{\partial\phi \cdots \partial\phi}\right|_{\phi_0}$

no vertices of degree 2

(no vertices of degree one because $\left.\frac{\partial S}{\partial\phi}\right|_{\phi_0} = 0$)

Discussion

Consider (WLOG) the case

$$\phi_0 = \phi_{min} = 0, \qquad S(\phi_0) = 0$$

By Taylor expansion with $A = \frac{\partial^2 S}{\partial\phi\partial\phi}\Big|_{\phi_0}$

$$S(\phi) = 0 + 0 + \tfrac{1}{2}(\phi, A\phi) + \underbrace{\text{remainder}}_{r(\phi) = O(\phi^3)}$$

$$\int e^{-\alpha S} d\phi = N(\alpha) \frac{1}{N(\alpha)} \int d\phi \; e^{-\frac{1}{2}(\phi, \alpha A \phi) - \alpha r(\phi)}$$

$$= N(\alpha) \int d\mu_C(\phi) \; e^{-\alpha r(\alpha^{-1/2}\phi)}$$

$\alpha r(\alpha^{-1/2}\phi) = \alpha^{-1/2} O(\phi^3)$ so it looks small as $\alpha \to \infty$.
Hence the Feynman expansion explains an important part of the Laplace expansion, but to derive the Laplace expansion properly you also need the idea of restricting the range of the integral to a small neighbourhood of ϕ_0 modulo $e^{-O(\alpha)}$ error. I am not going to explain that because the only role of the Laplace expansion in these lectures is to provide some background.

Reference: Vol 2 Hormander, also does oscillatory integrals. He does not bring in the connection with graphs but does show existence of asymptotic expansions in α^{-1}. Once you know they exist, identifying them with the prescriptions given here should be easy, but I do not know a good reference.

2.3 Ex (Stirling Formula)

To show that $\ln \dfrac{n!}{n^n e^{-n} \left(\frac{n}{2\pi}\right)^{-1/2}} \sim \dfrac{1}{12n}$ as $n \to \infty$

$$n! = \int_0^\infty t^n e^{-t}\, dt$$

$$= n^{n+1} \int_0^\infty e^{-n\, S(t)}\, dt \qquad (\text{by } t \to nt)$$

$S(t) = t - \ln t$

S	$S^{(2)}$	$S^{(3)}$	$S^{(4)}$
1	1	-2	6

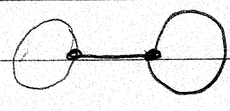
Minimum $t = 1$

so $C = 1$

(and n plays role of α)

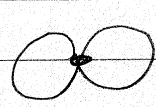
$$\ln \frac{n!}{n^n e^{-n} \left(\frac{n}{2\pi}\right)^{-1/2}} \sim$$

large deviations — Gaussian $N(n)$



$\left(\frac{1}{2}\right)^2 \frac{1}{2} (n^{-1}C)^3 (-nS^{(3)})^2$

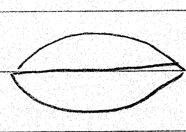
$+$



$\left(\frac{1}{2}\right)^2 \frac{1}{2} (n^{-1}C)^2 (-nS^{(4)})$

$+$

These are the $O(1/n)$ diagrams



$\left(\frac{1}{2}\right)^0 \frac{1}{2(3!)} (n^{-1}C)^3 (-nS^{(3)})^2$

$$= \frac{1}{12} n^{-1}$$

These are the only graphs whose vertices have degree ≥ 3 and which have # vertices − # edges $= -1$ $\rightsquigarrow$ Order(n^{-1})

Using Euler (Faces − Edges + Vertices) $\Rightarrow$ all have 2 cycles. Physicists call this the <u>loop expansion</u> because # cycles determines order in α^{-1}

2.4 Example (of Feynman expansion, not Laplace-Feynman)

$$P = \sum_i :(\phi_i' + S_i)^4:_{v'+C} / 4!$$

where ϕ_i' are parameters, $\text{Int} = \int d\mu_C(S)\, e^{-gP}$

$\ln(\text{Int}) =$  $\displaystyle -g \sum_i \frac{:\phi_i'^4:_{v'}}{4!}$

$+$  $\displaystyle g^2 \frac{1}{2} \sum_{i,j} \frac{:\phi_i'^3:_{v'}}{3!} C_{ij} \frac{:\phi_j'^3:_{v'}}{3!}$

$+$  $\displaystyle g^2 \frac{1}{2(2!)} \sum_{i,j} \frac{:\phi_i'^2:_{v'}}{2!} C_{ij}^2 \frac{:\phi_j'^2:_{v'}}{2!}$

$+$  $\displaystyle g^2 \frac{1}{2(3!)} \sum_{i,j} \phi_i' C_{ij}^3 \phi_j'$

$+$  $\displaystyle g^2 \frac{1}{2(4!)} \sum_{i,j} C_{ij}^4$

$+ O(g^3)$

Homework $\displaystyle :\frac{\phi_i^p}{p!}: \; :\frac{\phi_j^p}{p!}: = :\Big(e^{\Delta_{AB}} \underbrace{\frac{\phi_i^p}{p!}}_{A} \underbrace{\frac{\phi_j^p}{p!}}_{B}\Big):$ Use this to rewrite in Wick powers,

$$\ln \text{Int} = -g \sum \frac{:\phi_i'^4:}{4!} + g^2 \frac{1}{2(2!)} \sum_{i,j} (v_{ij}^2 - v_{ij}'^2) :\frac{\phi_i'^2}{2} \frac{\phi_j'^2}{2}:$$

$$+ \frac{g^2}{2(3!)} \sum (v_{ij}^3 - v_{ij}'^3) :\phi_i' \phi_j':$$

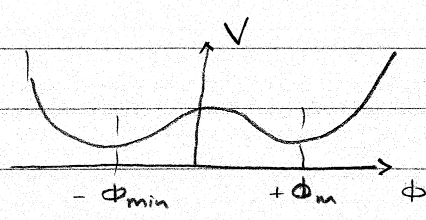
$$+ \frac{g^2}{2(4!)} \sum (v_{ij}^4 - v_{ij}'^4) + O(g^3)$$

for example, $v_{ij}^2 - v_{ij}'^2 = \underbrace{2 C_{ij} v_{ij}'}_{\text{from}}$  $+ \underbrace{C_{ij}^2}$ 

2.5 Ken Wilson [Wilson & Kogut]

In our discussion of the Ising Model,

$$S(\phi) = \frac{1}{2}(\nabla\phi, \nabla\phi) + \sum_{x\in\Lambda} V(\phi_x)$$



A Laplace type of expansion with $\phi_0 = \pm\phi_{min}$ is based on the intuition that $\phi \simeq \phi_0$, but when $\Lambda \subset \mathbb{Z}^d$ is very large, even if you fix $\phi = \phi_{min}$ at $\partial\Lambda$, fluctuations will make the Laplace expansion very non-uniform in Λ. (Physicists say "mean field theory does not take fluctuations into account")

Wilson had several crucial ideas; one of them is that it is better to do many small Laplace expansions as opposed to one big one. Thus

(*) [Gallavotti et al.]

if $v = C_1 + C_2 + \cdots + C_N$ and $\Phi \sim \mu_v$

then

$$\Phi = \zeta_1 + \underbrace{\zeta_2 + \cdots + \zeta_N}_{\Phi'} \qquad \zeta_i \sim \mu_{C_i} \text{ independent}$$

$$\mathbb{E}\, I^\Lambda = \mathbb{E}\left(\underbrace{\mathbb{E}(I^\Lambda \mid \zeta_2, \zeta_3, \ldots, \zeta_N)}\right)$$

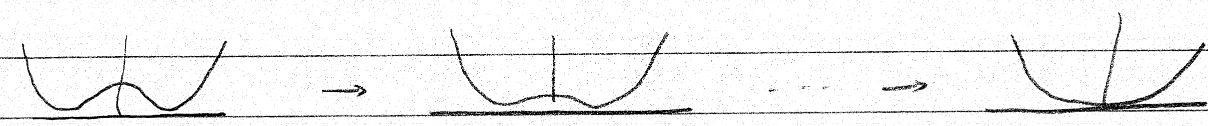
try to write this as $(I')^\Lambda$ with $I' = I'(\Phi')$

In Math Physics problem, find I' s.t

$$\int d\mu_{C_1}(\zeta_1)\; I^\Lambda(\Phi' + \zeta) = (I')^\Lambda(\Phi')$$

and then study $I \mapsto I' \to I'' \cdots$ etc

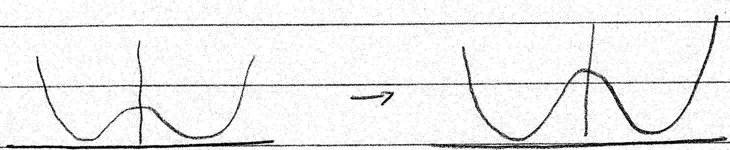
You might, in the Ising Model example, find, with $I = e^{-V}$ that V evolves as



High temp

critical

or



$\to$ 2 ergodic measures

low temp.

Summary want $I_0, I_1, \dots, I_N$ (subscript is a scale, not a lattice point)

$$E\left(I_j^{\wedge} \mid \zeta_{j+1}, \zeta_{j+2}, \dots\right) = I_{j+1}^{\wedge}$$

Backward martingale + factorisation

(It will not be possible to keep $I_{j,x}$ a function only of Φ'_x)

The only case where this program can be implemented almost exactly as described is hierarchical models and we should consider these next because they also illustrate the crucial role of scaling in in program, namely Wilson assumes that

$$\Phi'_x \overset{\mathcal{D}}{=} L^{-[\phi]} \Phi_{L^{-1}x}$$

parameter $[\phi] > 0$ called dimension

This is a restriction on the covariance C of Φ.

2.6 Hierarchical lattices [Dyson] [Steve Evans]

Λ^∞ is a countable Abelian group with

- $L^{-1}: \Lambda^\infty \to \Lambda^\infty$ homomorphism
- Ultrametric Norm such that $|x| = \begin{cases} L\,|L^{-1}x| \\ 0 \quad \text{if } L^{-1}x = 0 \end{cases}$

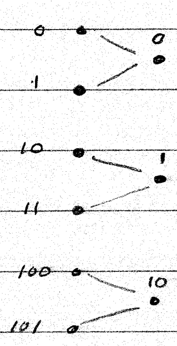
Ex $\Lambda^\infty = \{\text{finite binary sequences}\}$

$L^{-1} = 2^{-1} = \text{shift}$

$|x| = 2^{\text{length}(x)}$

Abelian Gp: add / subtract "without carrying"

$$\begin{array}{r} 11 \\ +\ \ 1 \\ \hline 10 \end{array}$$

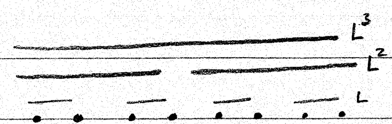


action of L^{-1}

Ultrametric: $|x+y| \leq \max(|x|, |y|)$

This means there are no overlapping balls: $B \cap B' \neq \phi \Rightarrow B \subset B'$ or $B' \subset B$

Balls are generated by repeated action of L^{-1}



In the example there are L^j points in ball of diameter L^j so the dimension is one. It is also possible to construct d dimensional hierarchical lattices for $d = 2, 3, \cdots$ using L^d-ary sequences

<u>Defn</u> If $(\Phi_x : x \in \Lambda^\infty)$, where Λ^∞ is hierarchical, satisfies

① $\Phi_x \overset{\mathcal{D}}{=} \zeta_x + \Phi'_x$ independent fields

② ζ_x, ζ_y independent if $|x-y| \geq L$

③ $\Phi'_x \overset{\mathcal{D}}{=} L^{-[\Phi]} \Phi_{L^{-1}x}$

we call it the <u>hierarchical</u> <u>(massless)</u> <u>Gaussian</u> of dimension $[\Phi]$.
[Brydges Evans Imbrie]

If we have ① and ② then we can solve

$$\mathbb{E}(I^\Lambda \mid \Phi') = (I')^{L^{-1}\Lambda}$$

(Note factorisation on next scale)

for I' because, by ②

$$\mathbb{E}(I^\Lambda \mid \Phi') = \prod_{B \in L^{-1}\Lambda} \mathbb{E}(I^B \mid \Phi')$$

so

$$I'_B = \mathbb{E}(I^B \mid \Phi')$$

<u>Notation</u>
$B \in L^{-1}\Lambda$
is being identified
with pre-image under L^{-1}
which is an L-ball in Λ.

③ enables us to consider I and I' as "in the same space" of functions of Φ so that for example we can ask if the map drives I to a fixed point.

if $I_x = e^{-g:\Phi_x^4:}$

$$I^B = e^{-g\sum_{x\in B} :\Phi_x^4:}$$

$$\mathbb{E}(I^B \mid \zeta_2, \dots, \zeta_N) = \int d\mu_{C_1}(\zeta_1)\, e^{-P} \qquad P \text{ as in Ex 2.4}$$

so by Ex 2.4

$$I'_B = e^{-g' :\Phi_B'^4:_{v'} - a' :\Phi_B'^2:_{v'} - b} \; e^{c' :\Phi_B'^6:_{v'} + O(g^3)}$$

where, since $\Phi'_x = \Phi'_y \; \forall x, y \in B$, we wrote $\Phi'_B = \Phi'_x$, and

$$g' = L^d\left(g - \beta g^2 + O(g^3)\right)$$

from $\sum_{x\in B}$

and if $C_{ij} = C_{|i-j|}$, $v_{ij} = v_{|i-j|}$, then from Ex 2.4,

$$\beta = \frac{1}{2(2!)} \sum_j \left(v_{ij}^2 - v_{ij}'^2\right)$$

To compare this with $I = I_0$ put in $\Phi'_x = L^{-[\phi]} \Phi_{L^{-1}x}$

leave the x index at Φ

$$I_0 = e^{-g_0 :\Phi^4:_v} \qquad g_0 = g$$

becomes

$$I_1 = e^{-g_1 :\Phi^4:_v - a_1 :\Phi^2:_v - b_1 - c_1 :\Phi^6:_v + O(g_0^3)}$$

(Yes $: :_{v'}$ scales to $: :_v$)

$$g_1 = L^{d-4[\phi]} \left(g_0 - \beta g_0^2 + O(g_0^3) \right)$$

In models such as 4 dimensional SAW, $[\phi] = 1$ when $d = 4$ (You will see why later), so

$$g_{j+1} = g_j - \beta g_j^2 + O(g_j^3)$$

which implies $g_j \sim \frac{1}{j}$ as $j \to \infty$.

To understand a_1, c_1 we should have started with an I_0 containing in the exponent $a_0 :\Phi^2: + c_0 :\Phi^6:$ as well as $g_0 :\Phi^4:$ then we would find that

$$c_{j+1} = \underline{\underline{L^{d-6[\phi]}}} c_j + O(g_j^2)$$

if $d - 6[\phi] < 0$, $c_j \simeq O(g_j^2)$ for all j if $c_0 = 0$

<u>Defn</u> : say that ϕ^p is <u>irrelevant</u> if $d - p[\phi] < 0$, <u>relevant</u> if $d - p[\phi] > 0$, <u>marginal</u> if $d = p[\phi]$.

The coefficients g_j, a_j, b_j, c_j are called (<u>running</u>) <u>coupling constants</u>

<u>Summary</u> According to perturbation theory all coupling constants in front of irrelevant $:\phi^p:$ remain $O(g_j^2)$ for all j if ϕ^4 is marginal. However relevant coupling constants will want to grow exponentially as $j \to \infty$.

In the preceding calculations we completely ignored the fact that ϕ^6 will make the integrals over ϕ divergent so we cannot really work with e^{ϕ^6} etc. These kinds of problems along with the fact that the Taylor expansions in powers of ϕ are only asymptotic, suggest

$$I = e^{-g:\phi^4: - a:\phi^2:} \left(1 + c:\phi^6: + r\right)$$

as a better way to control the hierarchical RG. Next lecture will be about a natural generalisation of this representation which is also good for Euclidean models.