Lecture I

Gaussian measures
(from the theoretical physics point of view)

Labels $i \in \Lambda$ (finite set)

variables $(\phi_{i \in \Lambda}) \in \mathbb{R}^\Lambda$

Lebesgue measure $d^\Lambda \phi$

A positive definite $\Lambda \times \Lambda$ matrix,

Recall A pos. def. if $(\phi, A\phi) > 0 \;\; \forall \phi \neq 0$ iff eigenvalues of A positive.

$$d\mu_C = \frac{1}{N} e^{-\frac{1}{2}(\phi, A\phi)} d^\Lambda \phi$$

$$\int d\mu_C \, \phi_i \phi_j = (A^{-1})_{ij} \; ; \qquad C \overset{\text{def}}{=} A^{-1}$$

$$\int d\mu_C \, e^{i(\phi, f)} = e^{-\frac{1}{2}(f, A^{-1} f)}$$

$$N = (2\pi)^{|\Lambda|/2} \det^{-\frac{1}{2}} A$$

Lemma For every pos. def. $\Lambda \times \Lambda$ matrix C there is a unique Gaussian prob. measure on $\mathbb{R}^\Lambda$ with covariance C.

Proof: Let $d\mu$ be as above with $A = C^{-1}$

<u>Main example</u> (massive Gaussian field, massive free field)

Λ is a graph (V,E)

Dirichlet form: $(\nabla\phi,\nabla\phi) \overset{\text{def}}{=} \sum_{ij\in E} (\phi_i-\phi_j)^2$

$$d\mu = \frac{1}{N} e^{-\frac{1}{2}(\nabla\phi,\nabla\phi) - \frac{1}{2}m^2(\phi,\phi)} d^\Lambda\phi$$

$m > 0$ is called the "mass"

The Laplacian Δ is the unique $\Lambda\times\Lambda$ matrix s.t.

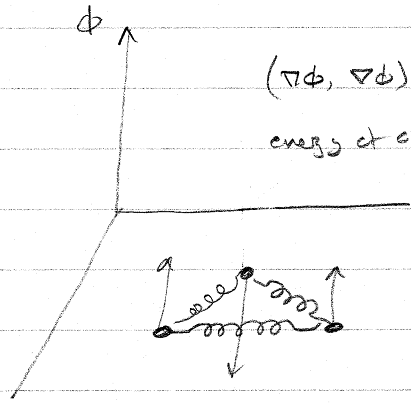
$$(\nabla\phi,\nabla\phi) = (\phi, -\Delta\phi)$$

so

$$C = (m^2 - \Delta)^{-1}$$

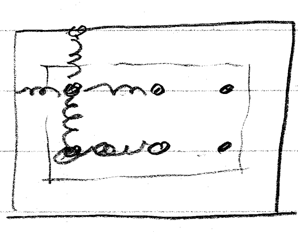
You cannot normalise this measure when $m=0$.

Think of the edges of graph as springs. A constant displacement at all nodes does not change $(\nabla\phi,\nabla\phi)$



$(\nabla\phi,\nabla\phi)$ is the potential energy of all the springs

<u>Harmonic</u>
<u>Bedsprings</u>
has energy $(\nabla\phi,\nabla\phi)$



$\phi = 0$ outside Λ

$\sum_{ij\in E}(\phi_i-\phi_j)^2$ with $\phi=0$ at $\partial\Lambda$

<u>Dirichlet</u> <u>b.c.s</u>

Properties of $C = m^2 - \Delta$ for (V, E)

(1) $m^2 C_{ij} \geq 0$

(2) $\sum_j m^2 C_{ij} = 1$

(3) $m^2 C_{ij} \leq C(\epsilon) \left(\frac{\nu}{m^2+\nu-\epsilon}\right)^{dist(i,j)}$, $\epsilon > 0$

ν = max degree

Proof of (1,3): $m^2 - \Delta = \underbrace{D}_{\text{diagonal}} - \underbrace{O}_{\text{off-diag.}}$

Hint: $(D-O)^{-1} = D^{-1} + D^{-1} O D^{-1} + D^{-1} O D^{-1} O D^{-1} + \cdots$

Entries of $-O$ correspond to cross terms in $(\nabla\phi, \nabla\phi)$ which negative coefficients so $O_{ij} \geq 0 \ \forall i \neq j$. $D_{ii} > 0 \ \forall i$

(2) Note that $\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}$ is an eigenvector for $-\Delta$ with eigenvalue zero.

Also
$$\sum_j C_{ij} = \left(C \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \right)_{i^{th}\text{ entry}}$$

Partition fn of Ising Model (Idea of Mark Kac, Siegert)

See PCMI notes

$$Z = \sum_{\sigma \in \{1,1\}^\Lambda} e^{\frac{1}{2}\sum_{i,j} J_{ij} \sigma_i \sigma_j}$$

Object: write it as Gaussian integral.

Assume J pos. def. $\exists!\, d\mu$ Gaussian s.t

$$= \sum_{\sigma} \int d\mu \; e^{\sum_i \phi_i \sigma_i}$$

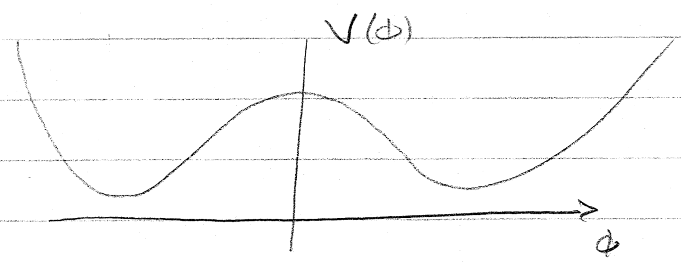
$$= \int d\mu \prod_{i \in \Lambda} \left(\sum_{\sigma_i} e^{\phi_i \sigma_i} \right)$$

$$= \int d\mu \prod_{i \in \Lambda} \cosh \phi_i$$

If $J = \beta (1 - \Delta)^{-1}$

$$= N^{-1} \int d^\Lambda \phi \; e^{-\frac{\beta}{2}(\nabla\phi, \nabla\phi) - \sum_i \overbrace{\left(\frac{\beta}{2}\phi_i^2 - \log\cosh\phi_i\right)}^{V(\phi_i)}}$$

Perturbation of massless Gaussian



<u>Wick Theorem</u> P any polynomial in ϕ, μ_C Gaussian,

$$\int d\mu_C \; P = e^{\frac{1}{2}\Delta_C} P \Big|_{\phi=0}$$

$$\Delta_C = \frac{1}{2} \sum_{i,j} C_{ij} \frac{\partial}{\partial \phi_i} \frac{\partial}{\partial \phi_j}$$

<u>Proof</u> Case $|\Lambda| = 1$

$$\mu_{tC} * P = \int e^{-\frac{\phi'^2}{2t}} P(\phi + \phi')\, d\phi'$$

is unique soln to

$$\frac{\partial u(t,\phi)}{\partial t} = \frac{\partial^2}{\partial \phi^2} u(t,\phi)$$

$$u(0,\phi) = P(\phi)$$

so is $u(t,\phi) = e^{t\, \partial^2 / \partial \phi^2} P(\phi)$. Therefore they are equal.

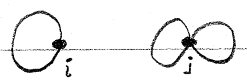
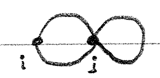
Set $t = \phi$, $\phi = 0$.

Feynman Diagrams

Example

$$\int d\mu_C \underbrace{\left(\frac{\phi_i^2}{2!}\;\frac{\phi_j^4}{4!}\right)}_{P} = e^{\frac{1}{2}\Delta_C}\, P\big|_{\phi=0}$$

$$= \frac{1}{3!}\left(\frac{1}{2}\Delta_C\right)^3 P\big|_0$$

$=$  $+$ 

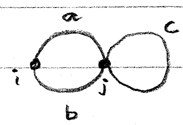
$$\left(\frac{1}{2}\right)^3 \underbrace{\frac{1}{2}}_{(*)} C_{ii}\, C_{jj}^2 \;+\; \left(\frac{1}{2}\right)\underbrace{\left(\frac{1}{2}\right)}_{(*)} C_{ij}^2\, C_{jj}$$

\# loops, also called tadpoles

(*) order of automorphism group

Defn A labelled graph

$G = ($Vertices, Edges, Incidence$)$

Ex 

$$\begin{array}{cc} & \begin{array}{cc} i & j \end{array} \\ I = & \left[\begin{array}{cc} 1 & 1 \\ 1 & 1 \\ 0 & 2 \end{array}\right] \begin{array}{c} a \\ b \\ c \end{array} \end{array}$$

$I_1 \sim I_2$ if $\exists$ permutations P_V, P_E such that

$$P_E I_1 P_V = I_2$$

An unlabelled graph is an equivalence class

In the lecture I explained the factors $\frac{1}{2}$ in terms of automorphisms as defined on the right but notice that I have labelled the vertices but not the edges in the example. By this I signal that we only want automorphisms where $P_V = I$.

Defn The Wick polynomial $:P: \; = e^{-\frac{1}{2}\Delta_C} P$

Ex $:\phi_i^4: \; = \phi_i^4 - \frac{1}{2}(4)(3)\, C_{ii}\,\phi_i^2 + \frac{1}{2}\frac{1}{2}\frac{1}{2}\, C_{ii}^2\; 4!$

Lemma if P, Q are monomials of different degree $\int d\mu_C :P: \, :Q: \; = 0$

Corollary If $d\mu_C$ is standard Gaussian, $C=1$, then $:\phi^p:$ are (monic) Hermite polynomials.

Proof of Lemma $\Delta = \Delta_C$, suppress C.

Leibniz rule: $\frac{\partial}{\partial\phi} = \frac{\partial}{\partial\phi_A} + \frac{\partial}{\partial\phi_B}$

$$e^{\frac{1}{2}\Delta} \quad A\,B$$

$$= e^{\frac{1}{2}(\Delta_{AA} + 2\Delta_{AB} + \Delta_{BB})} \quad A\, B$$

$$= e^{\Delta_{AB}} \left(\underbrace{e^{\frac{1}{2}\Delta_{AA}}}_{(*)} A\right) \left(e^{\frac{1}{2}\Delta_{BB}} B\right)$$

Replace A, B by $:P:$, $:Q:$ use def of Wick

$$= e^{\Delta_{PQ}} \quad P\,Q$$

$$= 0 \quad (\text{recalling we set } \phi = 0 \text{ at end})$$

(*) $e^{\frac{1}{2}\Delta_{AA}}$ generates $\left\{\begin{matrix}\text{tadpoles}\\ \text{loops}\end{matrix}\right.$ if $A = P$, but if $A = :P:$ it is cancelled and then no loops

Important: $\int d\mu \prod :\phi_i^{P_i}: = \Sigma$ graphs with no loops

Other properties:

$$\frac{\partial}{\partial\phi} :\phi^p: = p :\phi^{p-1}:$$

$$\int d\mu_{C_1}(s) \quad :(\phi+s)^p:_{C_1+C'} = :\phi^p:_{C'}$$

follow immediately from defn.

Example (cycles)

$$I_i \overset{\text{def}}{=} 1 + :\phi_i^2:/2!$$

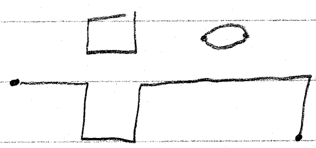
$$I^\Lambda \overset{\text{def}}{=} \prod_{i \in \Lambda} I_i$$

Then

$$\int d\mu_C \, I^\Lambda = \sum_{\substack{\text{sets } \{c_1, c_2, \ldots, c_n\} \\ \text{of disjoint cycles}}} \left(\prod_{ij \in \cup c_\alpha} C_{ij} \right) \left(\tfrac{1}{2}\right)^{\#\text{ of 2-cycles}}$$

(I forgot the factor $(\frac{1}{2})^{\#\text{ 2-cycles}}$ in the lecture)

A cycle $c = (i_1, i_2, \ldots, i_m) \in \Lambda^{\text{some } m}$ with the identification $c = c'$ if c' is a cyclic permutation or reversal of c.

$$\int d\mu_C \, I^{\Lambda \setminus \{a,b\}} \phi_a \phi_b = \sum$$ 

that is, a self avoiding walk from a to b together with cycles, all mutually disjoint.

The weight $\prod_{ij \in \cup c_\alpha} C_{ij}$ is not invariant under permutations of vertices, unless $C_{ij} = C_{k\ell}$ for all $i, j, k, \ell \in \Lambda$.
Therefore we should not expand in unlabelled graphs (because we need to know the labels to compute the weight)

Instead we expand into graphs with labeled vertices $i \in \Lambda$, unlabeled edges by $(\Delta = \Delta_C)$

$$\int d\mu_C \; I^\Lambda = e^{\frac{1}{2}\Delta \cdot} \; I^\Lambda \Big|_{\phi=0}$$

$$= \exp\Big(\sum_{\substack{ij \,\in \text{complete} \\ \text{graph on } \Lambda}} \Delta_{ij} \Big) \; \hat{I}^\Lambda \Big|_{\phi=0}$$

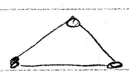
where $\Delta_{ij} = C_{ij} \, \partial/\partial\phi_i \, \partial/\partial\phi_j$, $\hat{I}_i = (1 + \frac{\phi_i^2}{2})$. The diagonal Δ_{ii} parts were absorbed by $I \to \hat{I}$, taking away the normal ordering.

$$= \prod_{\substack{ij \,\in \text{complete} \\ \text{graph}}} \Big(1 + \Delta_{ij} + \frac{\Delta_{ij}^2}{2}\Big) \; \hat{I}^\Lambda \Big|_{\phi=0}$$

The product is over choices no edge, 1 edge, 2 edges containing i and j for each pair of vertices $\{i,j\}$. It should now be clear how this gives the formula for $\int d\mu \, I^\Lambda$ in terms of cycles.

<u>Why 2-cycles have $\frac{1}{2}$:</u> This was an expansion into graphs with labeled vertices and $\frac{1}{|\mathrm{Auto}|}$ means divide by the order of the edge automorphism group

$|\text{Edge automorphisms}| = 2$



No edge automorphisms (except Identity) for m-cycles, $m \geq 3$.