Fluctuation estimates for sub-quadratic gradient field actions

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In this article we estimate fluctuations of the scalar field $\phi$ for a special class of sub-quadratic actions which grow like $|\nabla \phi|^{2\alpha}$, $0 < \alpha < 1$. In particular if $\alpha = 1/2$ we show that in three dimensions $\langle e^{\gamma \phi} \rangle$ is bounded for $\gamma$ small. For each edge $(jk)$ we introduce an auxiliary field $t_{jk} \in \mathbb{R}$ to express the action as a superposition of Gaussian free fields. The effective action which arises from integrating over the Gaussian field is shown to be convex in $t$. The Brascamp-Lieb inequality is then applied to obtain the desired estimates on a nonuniformly elliptic Green’s function.

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Dedicated to Elliott Lieb on the occasion of his 80th birthday

The theory of convex functions of the gradient of a scalar field is well developed and applies to problems arising from anharmonic crystals, dipole gases, and random surfaces. See for example Refs. 4 and 6. Let $\phi_j \in \mathbb{R}$ with $j \in \Lambda \subset \mathbb{Z}^d$. In this article we consider a class of models described by finite volume measures which take the form

\begin{equation}
\text{(Normalisation)} \quad e^{-\sum_j (V(\nabla \phi_j) + \epsilon \phi_j^2)} \prod_{j \in \Lambda} d\phi_j, \tag{1}
\end{equation}

where $\epsilon > 0$ and for $\nabla \phi$ large,

\begin{equation}
V(\nabla \phi_j) = O \left((\nabla \phi_j)^{2\alpha}\right). \tag{2}
\end{equation}

We will state results for the case of periodic boundary conditions on $\Lambda$ and obtain estimates uniform in $\epsilon > 0$ as $\Lambda \to \mathbb{Z}^d$.

There does not seem to be any mathematical literature which addresses models of this form with $0 < \alpha < 1$. We are especially interested in large deviations of the field, given by $\langle e^{\gamma \phi} \rangle$ when $\alpha = 1/2$ because a similar action appears in the measure describing linearly edge reinforced random walks (ERRW). See for example Ref. 9. For a class of models described below, we also bound moments of $\phi$ for any $0 < \alpha < 1/2$, in which case the action is not convex. We use the word action to denote the function in the exponent of a weight that multiplies Lebesgue measure, which in this case is $\sum_j V(\nabla \phi_j)$.

To define our model let $\Lambda$ be a lattice torus $\mathbb{Z}_N^d$, where $N \geq 3$ is the side of the torus, which is an integer. In fact we make no use of symmetries. Let $\{j, k\}$ be an unordered pair of nearest neighbor sites in $\Lambda$. We denote such pair by $jk = \{j, k\}$ and let $E$ be the set of all such pairs. For each $jk \in E$ we introduce an auxiliary field $t_{jk} \in \mathbb{R}$ and we define a Gaussian action,

\begin{equation}
A(\phi, t) = \sum_{jk \in E} \left(1 + \frac{1}{2} \beta (\phi_j - \phi_k)^2\right) e^{\epsilon t} + \frac{1}{2} \sum_{j \in \Lambda} \epsilon \phi_j^2, \quad \beta, \epsilon > 0. \tag{3}
\end{equation}
Then the partition function for our model is defined to be a superposition of Gaussians,
\[
Z_{\Lambda}(\beta) = \int e^{-A(\phi, t)} \prod_{j \in E} \left( e^{-\epsilon v_{jk} t} + \frac{1}{2} \phi_j t \right) \prod_{j \in \Lambda} d\phi_j, \quad \kappa > 0.
\] (4)

Let \( \langle \cdot \rangle_\beta \) denote the expectation with respect to the probability measure on the space of \((\phi, t)\) configurations that is defined by the weight in the partition function (4). It depends on \( \Lambda \) and \( \epsilon \) but since our estimates are uniform in \( \Lambda \) and \( \epsilon \) we do not make these dependences explicit.

For \( \kappa = 1 \) the integral over \( t = (t_{jk}) \) can be explicitly evaluated (see (24) below) and the result, after dropping some factors of \( \pi \), is
\[
Z_{\Lambda}(\beta) = \int e^{-2\sum_{j \in E} \sqrt{1+12\beta(\phi_j - \phi_k)^2 + 2\sum_{j, k, m} \epsilon \phi_j^2}} \prod_{j \in \Lambda} d\phi_j,
\] (5)

which is a model as in (1) with \( \alpha = \frac{1}{2} \). For \( \kappa \neq 1 \) we cannot evaluate the \( t \) integral explicitly, but by considering the \( t_{jk} \) that maximizes the exponent it is not hard to see that integration over \( t \) produces a model as in (1) such that \( V(\nabla \phi_j) \) grows as \((\beta(\nabla \phi_j)^2)^{c(1 + \kappa)}\) and hence \( \alpha = \kappa/(1 + \kappa) \).

Nonconvex potentials have also been studied by superposition in Ref. 5. In this case the potentials correspond to \( \alpha = 1 \), because the corresponding \( t \) variables have compact support. Furthermore, the superposition is uniformly elliptic whereas ours is not, and this enables us to get \( \alpha < 1 \).

Let \( \{ v, w \} = \sum v_j w_j \) denote the scalar product. Define the finite difference elliptic operator \( D_\beta(t) \) by the quadratic form
\[
[f; D_\beta(t)f] = \beta \sum_{j \in E} (f_j - f_k)^2 / \epsilon + \epsilon \sum f_j^2
\] (6)

and let \( G(t; j, k) \) denote its Green’s function \( G(t) = (D_\beta(t))^{-1} \). Define \( G_0 = (-\beta \Delta + \epsilon)^{-1} = D_0^{-1} \) by setting \( t = 0 \).

The \( \phi \) field may be integrated out and, dropping factors of \( \pi \), we obtain
\[
Z_t = \int \det[D_\beta(t)]^{1/2} \prod_{j \in \Lambda} \left( e^{-\epsilon v_{jk} t} e^{-\epsilon v_{jk} t} + \frac{1}{2} \phi_j t \right) d\phi_j.
\] (7)

Following terminology that is standard in physics we call this an effective partition function. Note
\[
\langle \phi_0^2 \rangle = \langle G(t; 0, 0) \rangle_t, \quad \langle \epsilon^{\gamma} \phi_0 \rangle = \langle \epsilon^{\gamma} G(t; 0, 0) / 2 \rangle_t,
\] (8)

where the \( \langle \cdot \rangle_t \) denotes the expectation over the auxiliary \( t \) fields. Defining \( \phi \cdot v = \sum_j \phi_j v_j \) we have the slightly more general
\[
\langle (\phi \cdot v)^{2n} \rangle = (2n - 1)!! \langle (v; G(t)v)^n \rangle_t, \quad \langle \epsilon^{\gamma} \phi \cdot v \rangle = \langle \epsilon^{\gamma} (v; G(t)v)^n / 2 \rangle_t.
\] (9)

Lemma 1: For all \( t \) and \( \beta > 0 \), \( \ln \det[D_\beta(t)] \) is a convex function of \( t \).

This lemma is a key element in our proofs. A similar determinant arises in the hyperbolic sigma model studied in Ref. 10 where partial results on convexity were proved. Lemma 1 follows directly from Kirchoff’s theorem,\(^8\) also known as the matrix tree theorem. The version we use can be found in Theorem 1 of Ref. 1. See also Theorem 1.19 of Ref. 7. The matrix tree theorem expresses \( \det D(t) \) as a sum over weighted rooted forests with each nearest neighbor edge \((j, k)\) assigned a weight \( e^{\epsilon \phi} \) and each root a weight \( \epsilon \). Since \( \det D(t) \) is a positive superposition of exponentials its logarithm is convex.

Lemma 1 implies that the effective action in the \( t \) variables is convex. In fact the Hessian \( H \) is bounded from below by the diagonal
\[
\inf \left( e^t + \kappa^2 e^{-\kappa t} \right) \delta_{jk, lm} = c(\kappa) \delta_{jk, lm},
\] (10)

where this defines the positive constant \( c(\kappa) \).
Lemma 2: Let $v$ denote a vector in $R^\Lambda$ orthogonal to constants. The Green’s function satisfies the quadratic form bound

$$0 \leq [v; G(t)v] \leq \sum_{j \in E} ((G_0v)_j - (G_0v)_k)^2 e^{-t/k}.$$ (11)

Proof: Without loss of generality we take $\beta = 1$. We can also set $\epsilon = 0$ because $[v; G(t)v]$ increases as $\epsilon \downarrow 0$. Let $D_0 = \nabla^* \nabla = G_0^{-1}$. According to (6) we can write $D_{p,t} = \nabla^* A^2 \nabla$ where $A$ is the diagonal matrix whose entries on the diagonal are $e^{t/k}$. By integration by parts followed by the Schwartz inequality and the Brascamp-Lieb bounds.3

Since the action corresponding to $t_j, \phi_j$ is bounded by a constant $b$ assuming that $[v; G_0v]$ is finite. In the proof of this second result we will prove that $\langle \exp(\epsilon^{-1} t) \rangle < \infty$ for suitable values of $a$. As in Ref. 10 the method is a combination of a Ward identity and the Brascamp-Lieb bounds.3

Proposition 3: Let $\langle \cdot \rangle$ denote the expectation in $t$ and $\phi$ defined via (4). For $\lambda, \kappa > 0$, there is a constant $C(\lambda, \kappa)$ such that

$$\langle e^{-\lambda t_j} \rangle \leq C(\lambda, \kappa)$$ (12)

and $\langle (\phi \cdot v)^2 \rangle$ is bounded by constants $\overline{C}(\kappa, p)$ provided $[v; G_0v]$ is finite.

Proof: For $\lambda$ positive or negative, let $q(\lambda) = \ln(\langle e^{-\lambda t_j} \rangle)$. We bound $q(\lambda)$ using Taylor’s theorem to second order in $\lambda$. To do this let $F_\lambda = e^{-\lambda t_j}$ and let $q(\lambda) = \langle \cdot \rangle_\lambda = \langle \cdot \rangle/(F_\lambda)$. Then

$$q(0) = 0, \quad q'(0) = (-t_j), \quad q''(\lambda) = \langle (t_j - \langle t_j \rangle_\lambda)^2 \rangle_\lambda.$$ (13)

Since the action corresponding to $\langle \cdot \rangle_\lambda$ is also convex with Hessian bounded below as in (10), $q''(\lambda)$ is bounded by a constant $c(\kappa)$ using the Brascamp-Lieb bound [Ref. 3, Theorem 5.1]. By the bound and the Taylor expansion we have

$$\langle e^{-\lambda t_j} \rangle \leq e^{c(\kappa)\lambda^2/2} e^\lambda (-t_j).$$ (14)

To obtain a bound on $\langle -t_j \rangle$ we use a Ward identity generated by the change of variables

$$t_j \rightarrow t_j + b.$$ (15)

Since the partition function does not depend on the constant $b$, the derivative with respect to $b$ evaluated at $b = 0$ vanishes hence,

$$\left\langle -e^{t_j} \left[ 1 + \frac{1}{2} (\phi_j - \phi_k)^2 \right] + \kappa e^{-\lambda t_j} - \frac{1}{2} \right\rangle = 0.$$ (16)

Referring to (6), by $[f; Gf] = \sup_{\phi} \left( 2 [f; \phi] - [\phi; D\phi] \right)$ and $[\phi; D\phi] \geq \beta [e^{t_j} (\phi_j - \phi_k)^2]$, we have $\langle (\phi_j - \phi_k)^2 \rangle \leq \langle (\delta_j - \delta_k); G(t)(\delta_j - \delta_k) \rangle \leq \frac{1}{\beta} e^{-t_j}$, thus

$$\langle \kappa e^{-\lambda t_j} \rangle \leq \langle e^{t_j} \rangle + \frac{1}{2} + \frac{1}{2\beta}.$$ (17)
Jensen’s inequality implies \( \kappa e^{-\kappa(t_{jk})} \leq \langle e^{t_{jk}} \rangle + \frac{1}{2} + \frac{1}{2\beta} \) and by (14) with \( \lambda = -1 \),

\[
\kappa e^{-\kappa(t_{jk})} \leq e^{\kappa(t_{jk}) - 1/2} e^{\kappa(t_{jk})} \leq \frac{1}{2} + \frac{1}{2\beta}.
\]  

(18)

Therefore, for some positive constant \( C(\kappa) \),

\[
\langle -t_{jk} \rangle \leq C(\kappa).
\]  

(19)

We insert this bound into (14) with \( \lambda > 0 \) and obtain (12).

To bound the moments we use (9), Lemma 2 and the H"older inequality to bound the expectation of a product of \( e^{-t_{jk}} \) factors by (12).

\[ \text{Theorem 4: Let } \langle \cdot \rangle_{\beta} \text{ be the expectation defined via (4) and let } \kappa = 1 \text{ so that we are considering the model (5). If } v \text{ is orthogonal to constants and if } y^2[v; G_0v] < 1, \text{ then } \langle e^{y\Phi \cdot e} \rangle_{\beta} \text{ is uniformly bounded in } \epsilon \text{ and } \Lambda. \]

\[ \text{Proof: By (9) and Lemma 2 it suffices to estimate} \]

\[
\langle \exp(\frac{1}{2}y^2 \sum_{jk \in E} a_{jk} e^{-t_{jk}}) \rangle, \text{ with } a_{jk} = [G_0(j, v) - G_0(k, v)]^2.
\]  

(20)

Note that \( \sum_{jk \in E} a_{jk} = [v; G_0v] \leq 1 \). For \( \lambda \in [0, \frac{1}{2}y^2] \) let \( F_\lambda = \exp(\lambda \sum_{jk} a_{jk} e^{-t_{jk}}) \) and let \( g(\lambda) = \ln \langle F_\lambda \rangle \). Expand \( g \) to second order in \( \lambda \) using the principle in (13) to express the derivatives with respect to \( \lambda \) in terms of a new expectation, \( \langle \cdot \rangle_\lambda = \langle \cdot F_\lambda \rangle / \langle F_\lambda \rangle \). The effect of the additional factor \( F_\lambda \) reduces the convexity of the action expressed in (10) so that the Hessian corresponding to \( \langle \cdot \rangle_\lambda \) is now bounded below by the diagonal quadratic form

\[
(e^{t_{jk}} + e^{-t_{jk}} - \lambda a_{jk} e^{-t_{jk}})g_{jk,lm}.
\]  

(21)

These diagonal entries are bounded below by \( (1 - \lambda a_{jk})(e^{t_{jk}} + e^{-t_{jk}}) \) and \( \lambda a_{jk} \leq \frac{1}{2}y^2[v; G_0v] \leq 1/2 \). From the Brascamp-Lieb bound [Ref. 3, Theorem 4.1], (21) and \( \lambda a_{jk} \leq \frac{1}{2} \),

\[
0 \leq g''(\lambda) = \langle \sum_{jk} a_{jk} e^{-t_{jk}}; \sum_{jk} a_{jk} e^{-t_{jk}} \rangle_{\lambda} \\
\leq \sum_{jk} a_{jk}^2 \langle e^{-t_{jk}} [e^{t_{jk}} + e^{-t_{jk}} - \lambda a_{jk} e^{-t_{jk}}]^{-1} e^{-t_{jk}} \rangle_{\lambda} \\
\leq \sum_{jk} a_{jk}^2 (1 - \lambda a_{jk})^{-1} \langle e^{-t_{jk}} \rangle_{\lambda} \leq 2 \sum_{jk} a_{jk}^2 \langle e^{-t_{jk}} \rangle_{\lambda}.
\]  

(22)

Notice that the lower bound says that \( g'(\lambda) \) is increasing. From this bound on \( g''(\lambda) \) we get a bound on \( g'(\lambda) = \sum_{jk \in E} a_{jk} \langle e^{-t_{jk}} \rangle_{\lambda} \) by integrating \( g'' \) with respect to \( \lambda \):

\[
0 \leq g'(\lambda) = g'(0) + 2 \int_0^\lambda \sum_{jk \in E} a_{jk}^2 \langle e^{-t_{jk}} \rangle_{\lambda} \, d\lambda \\
\leq g'(0) + 2[v; G_0v] \int_0^\lambda \sum_{jk \in E} a_{jk} \langle e^{-t_{jk}} \rangle_{\lambda} \, d\lambda = g'(0) + 2[v; G_0v] \int_0^\lambda g'(s) \, ds \\
\leq g'(0) + 2[v; G_0v] \lambda g'(\lambda).
\]  

(23)

By the hypothesis and \( \lambda < \frac{1}{2}y^2 \) we can solve this inequality to obtain \( 0 \leq g'(\lambda) \leq (1 - 2\lambda[v; G_0v])^{-1} g'(0) \). Note that by Proposition 3, \( 0 \leq g'(0) \leq C(1, 1) [v; G_0v] \). Then we integrate this bound over \( \lambda \in [0, \frac{1}{2}y^2] \) and obtain the desired result.

\[ \text{Proof of (5). We obtained the model (5) from the integral formula} \]

\[
e^{-2z^2} = \frac{1}{\sqrt{\pi}} \int e^{-t} e^{-t^{1/2}} dt, \quad z > 0.
\]  

(24)
with
\[ z = 1 + \frac{1}{2} \beta (\phi_j - \phi_k)^2. \]  

(25)

To prove (24) we use the changes of variables \( t \mapsto t - \ln \sqrt{z} \) and \( u = e^t - e^{-t}, \)

\[
\int e^{-ze^t} e^{-e^{-t}-t/2} dt = \int e^{-\sqrt{z}e^t} e^{-\sqrt{z}e^{-t}} e^{-t/2} \sqrt{z} dt
\]

\[
= \frac{1}{2} \int e^{-\sqrt{z}(e^t + e^{-t})} (e^{t/2} + e^{-t/2}) \sqrt{z} dt,
\]

\[
= \frac{1}{2} \int e^{-\sqrt{z}e^{t/2} - e^{-t/2}} - 2\sqrt{z} (e^{t/2} + e^{-t/2}) \sqrt{z} dt
\]

\[
= \int e^{-\sqrt{z}u^2} - 2\sqrt{z} e^{-\sqrt{z}u^2} du = \sqrt{\pi} e^{-2\sqrt{z}}
\]

(26)

(27)

More general functions besides the square root can be represented as \( t \) integrals over Gaussians.

A function \( f : (0, \infty) \rightarrow \mathbb{R} \) is called completely monotonic if it is \( C^\infty \) and \((-1)^n f^{(n)}(x) \geq 0\), for \( n \geq 0, x > 0 \). A function \( f : (0, \infty) \rightarrow [0, \infty) \) is called a Laplace exponent, if it is \( C^\infty \) and \( f' \) is completely monotonic. If \( f \) is a Laplace exponent then \( e^{-tf(z)} \) can be written as an integral over \( s \in [0, \infty) \) of \( e^{sf(z)} \) with respect to a positive measure that depends on \( t \). Laplace exponents include \( f(z) = z^{\alpha}, \) but for \( \alpha \neq \frac{1}{2} \) we do not know if the weight in the integral representation is log concave as required by the Brascamp-Lieb bounds. See Section IX.11 of Ref. 11 and Ref. 2 for more background.

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