Outline

- Phase line - how to extract information from an equation without solving it:

  - steady states,

  - stability,

  - general “shape” of solutions.

- Equations for motion at low Reynolds number.

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- Draw the phase plane and sketch several solutions for the differential equation

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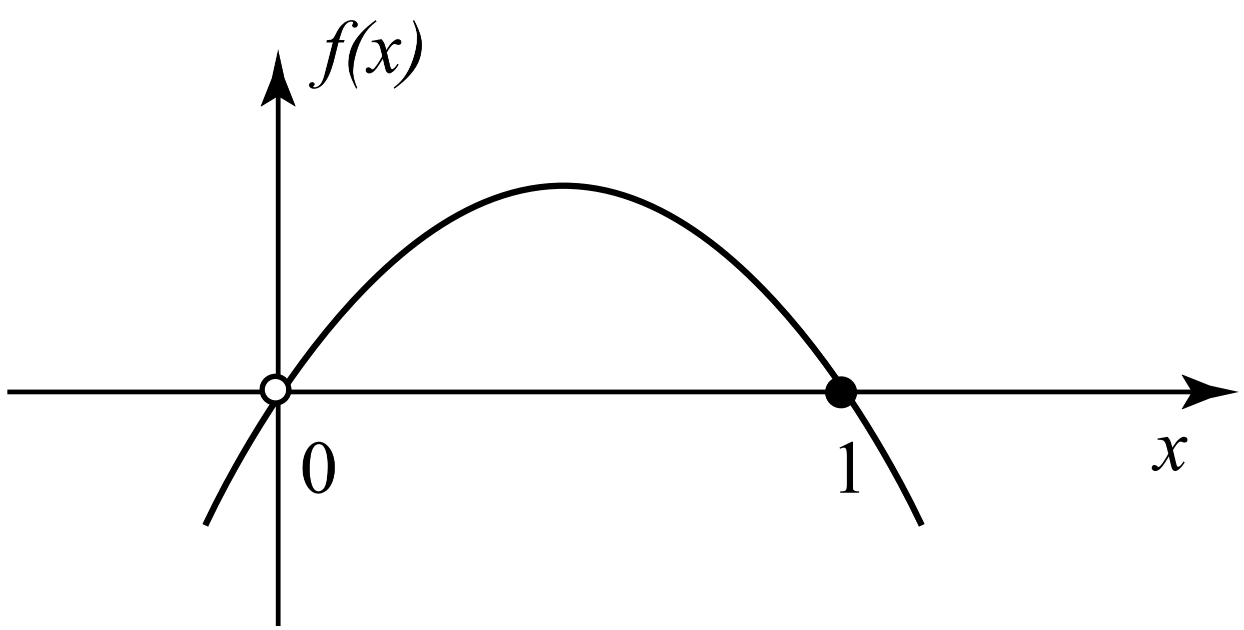

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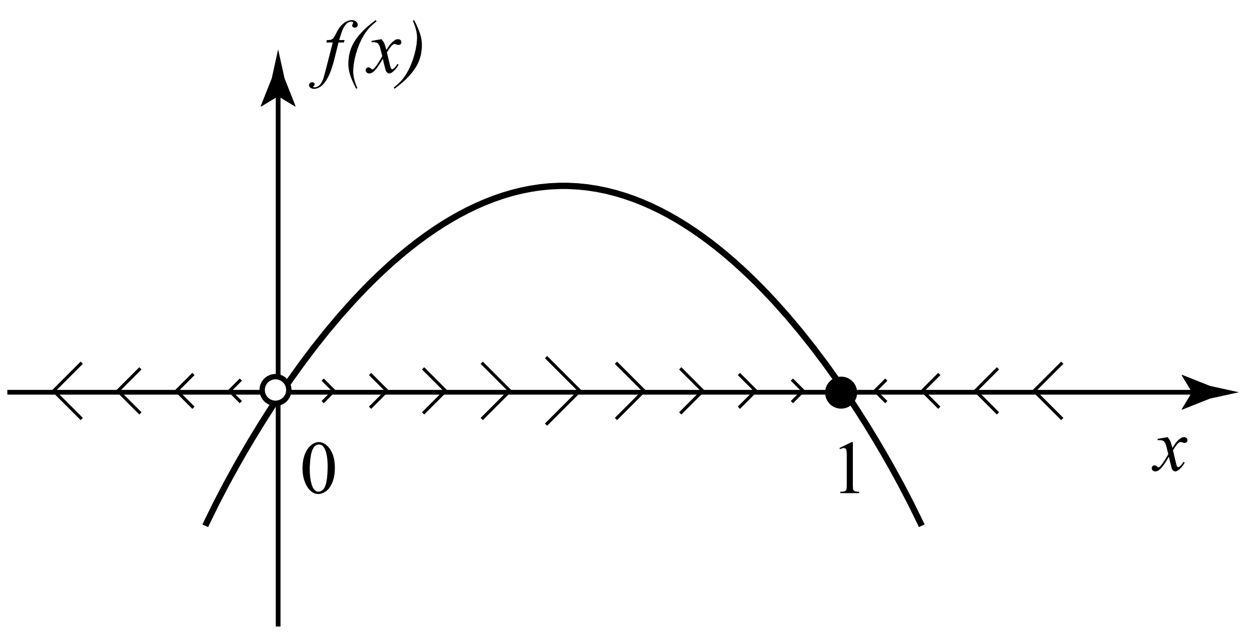

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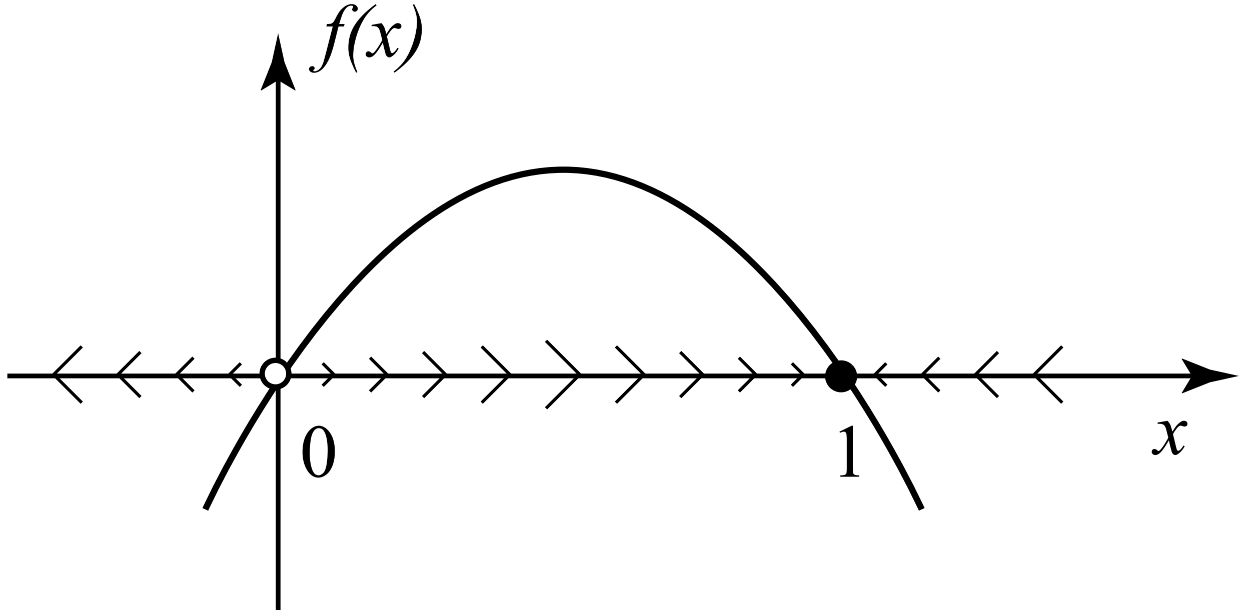

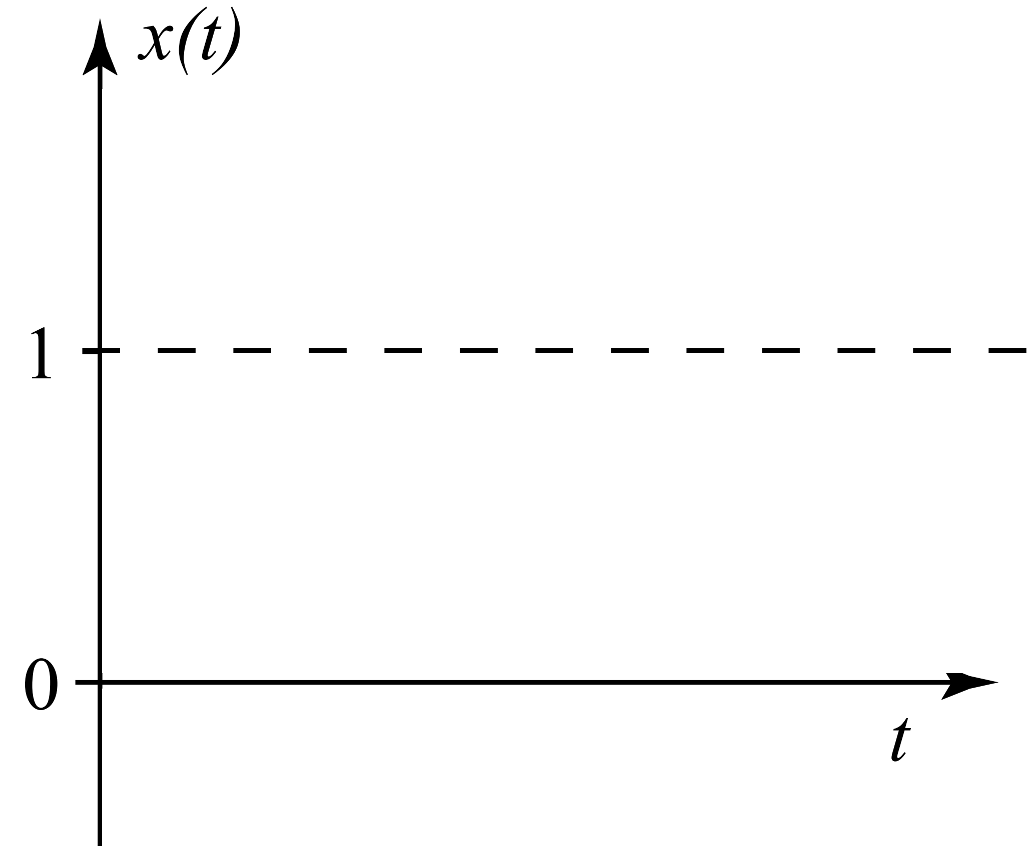

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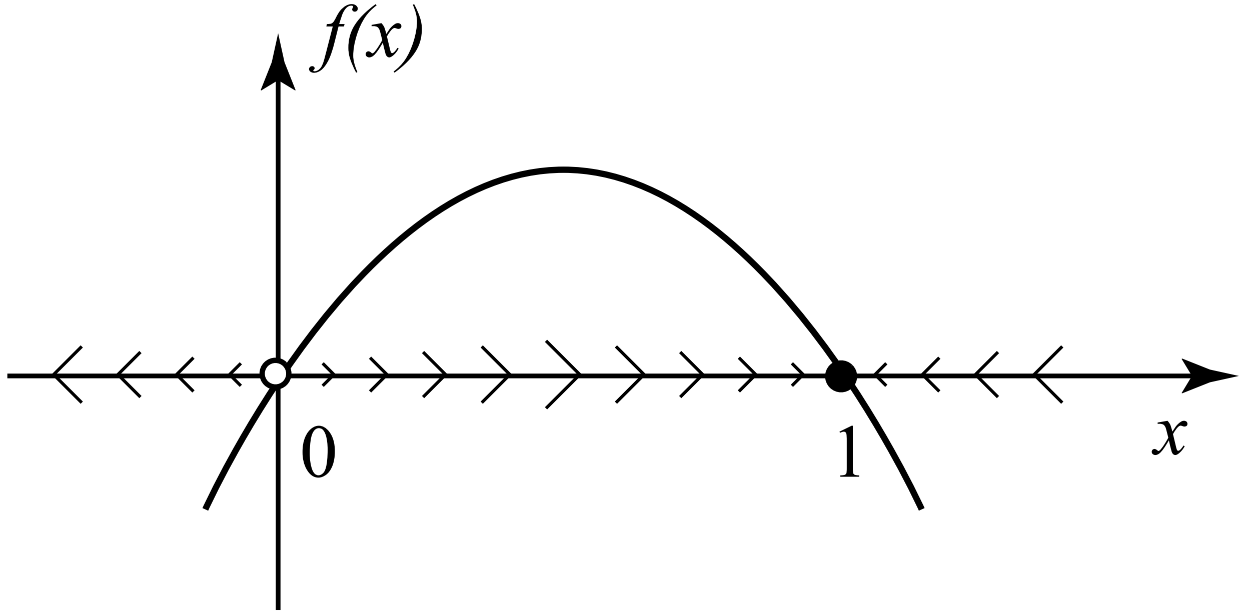

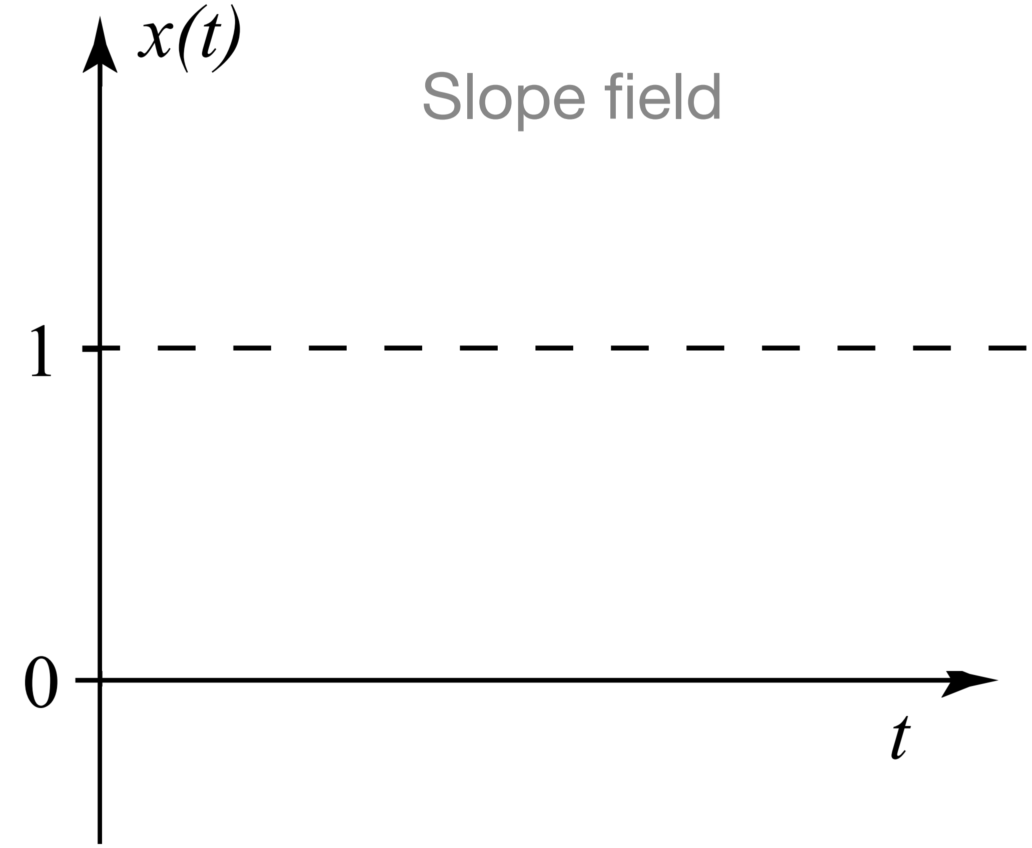

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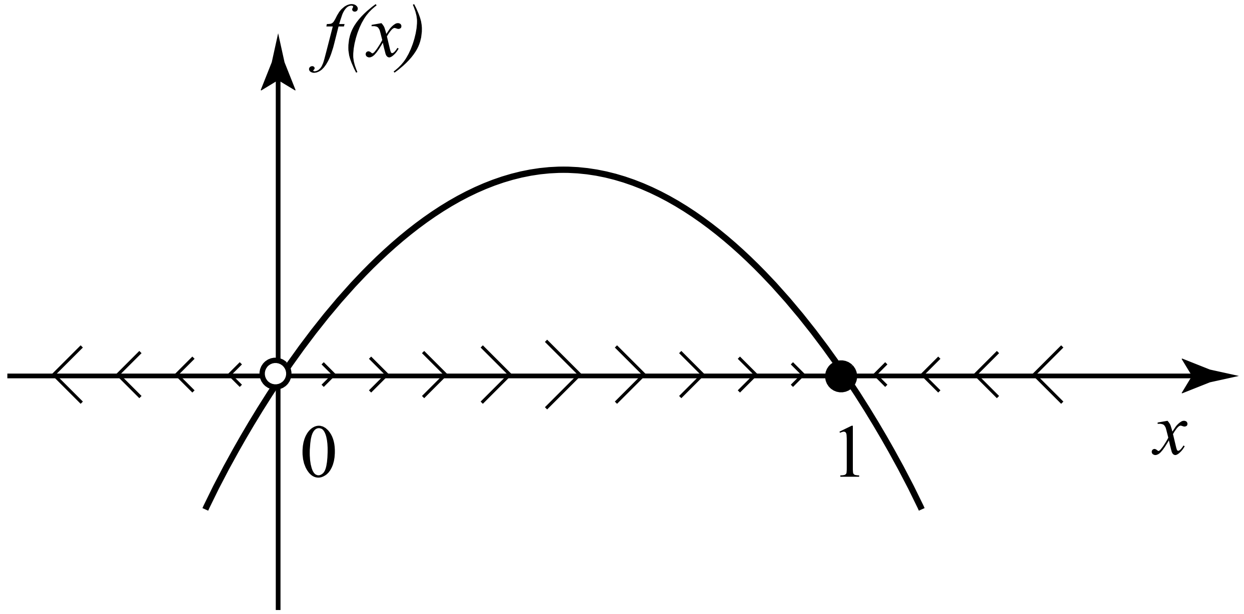

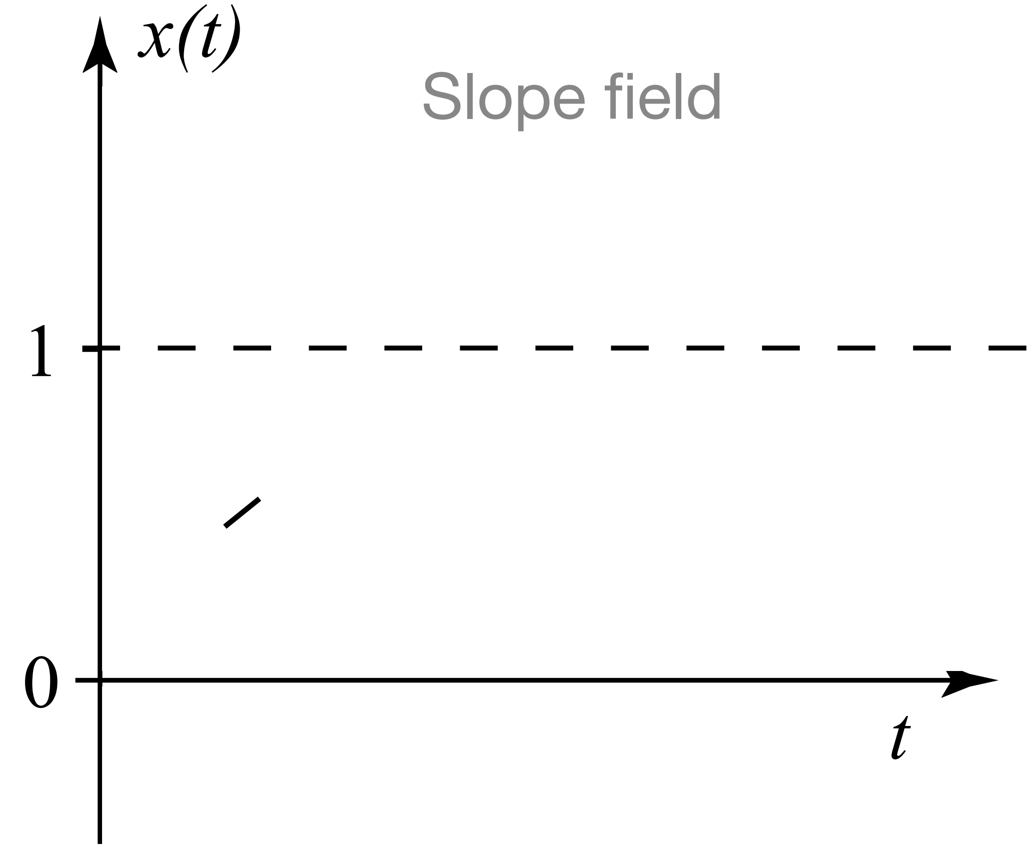

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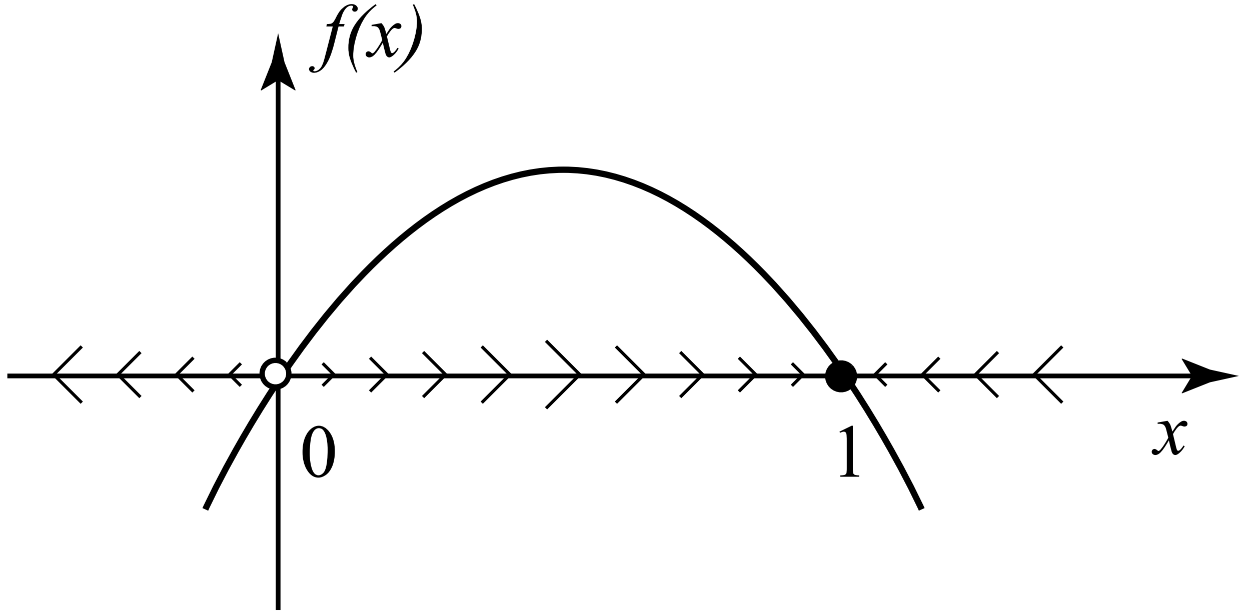

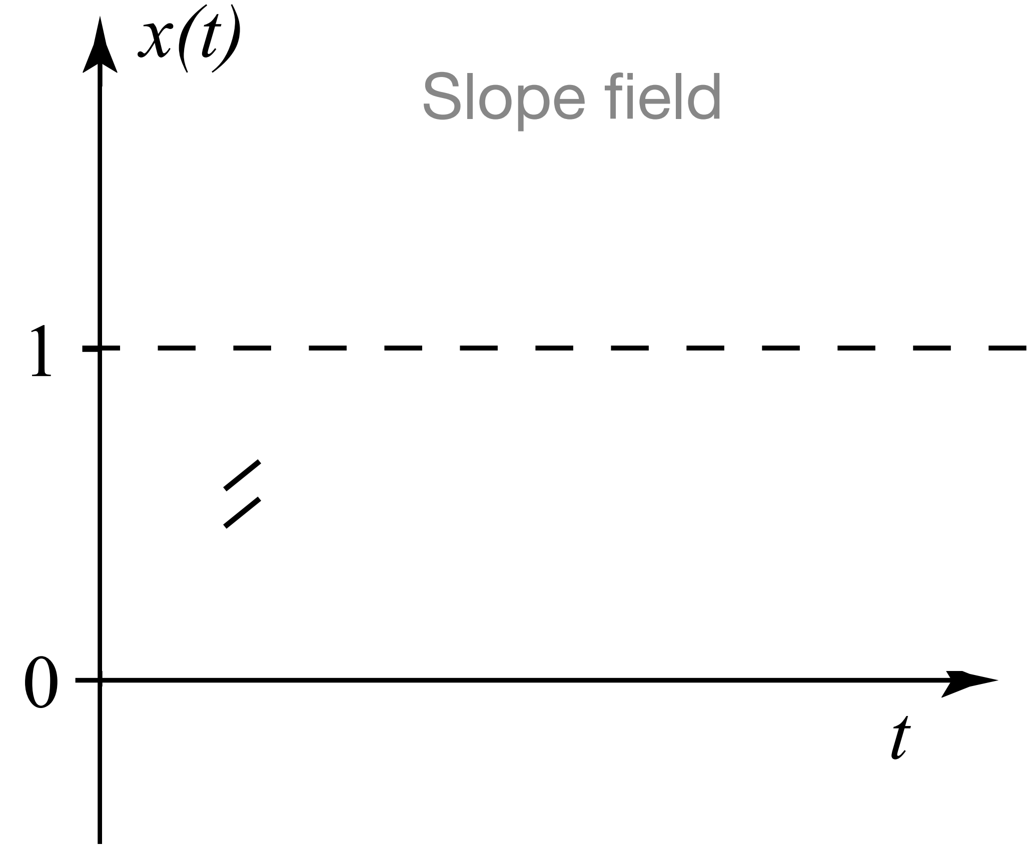

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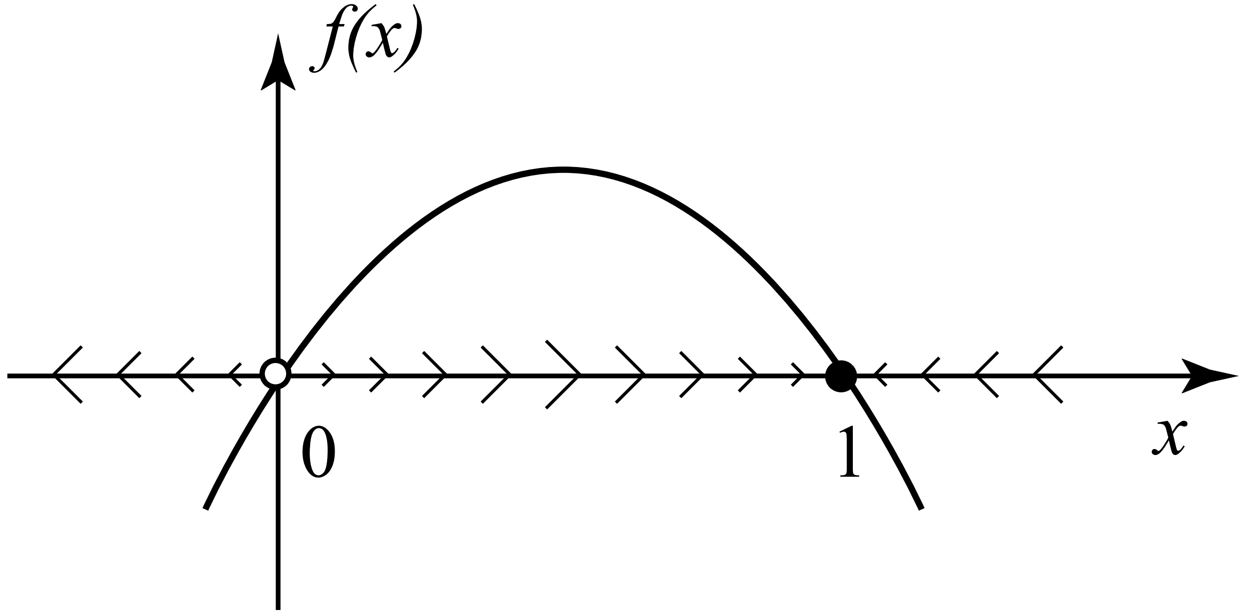

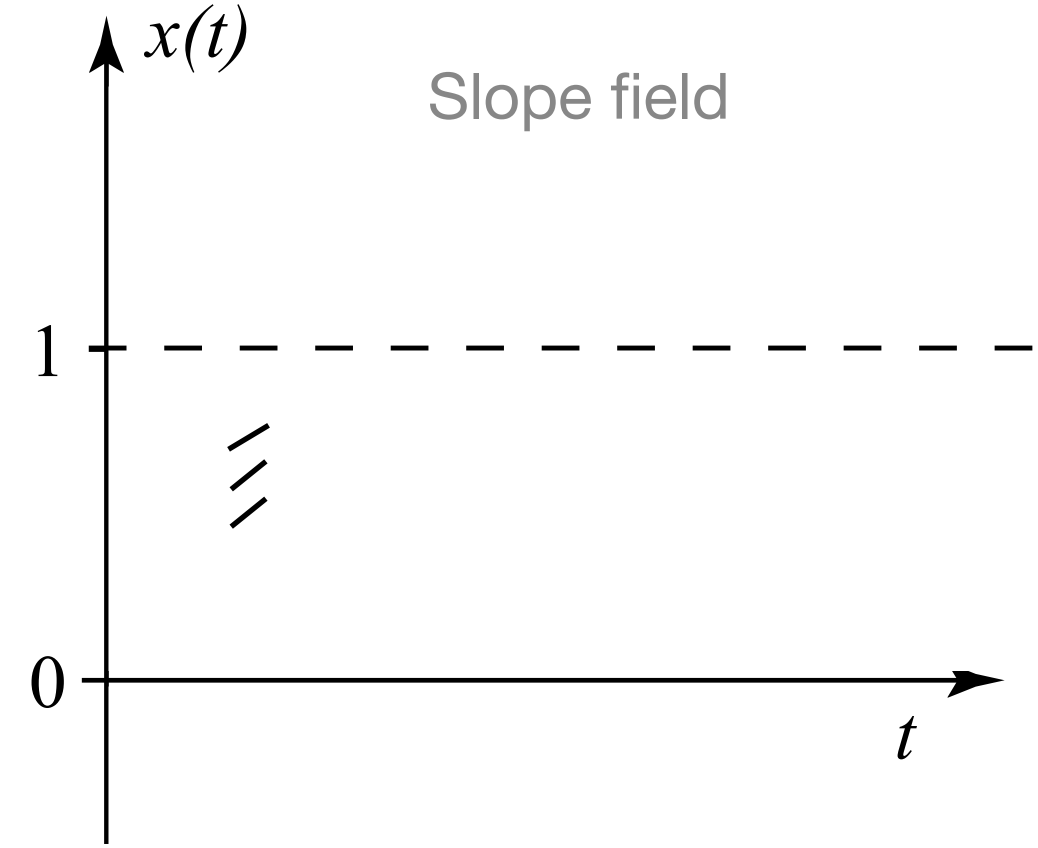

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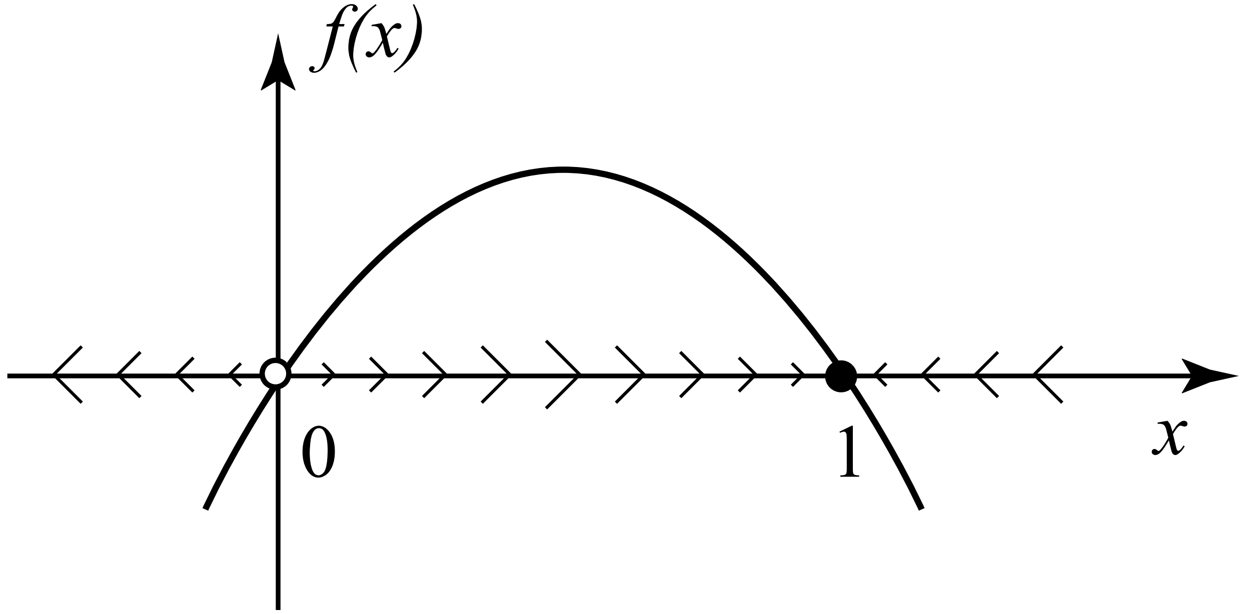

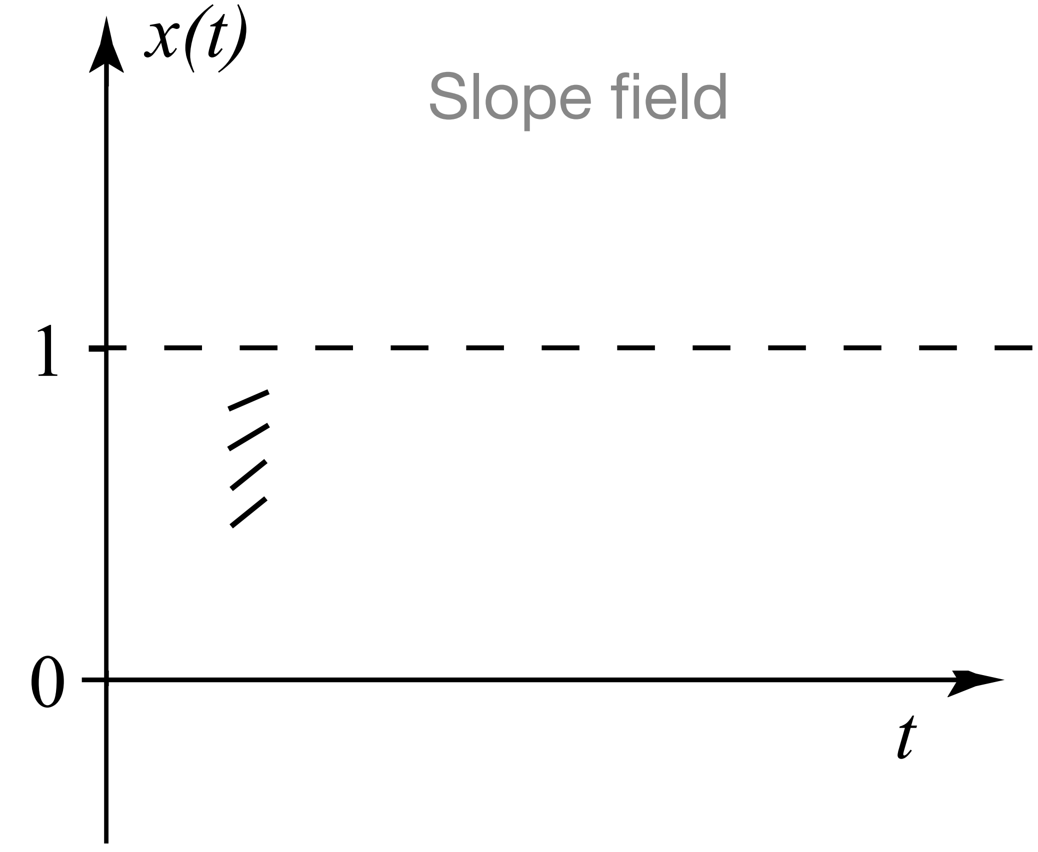

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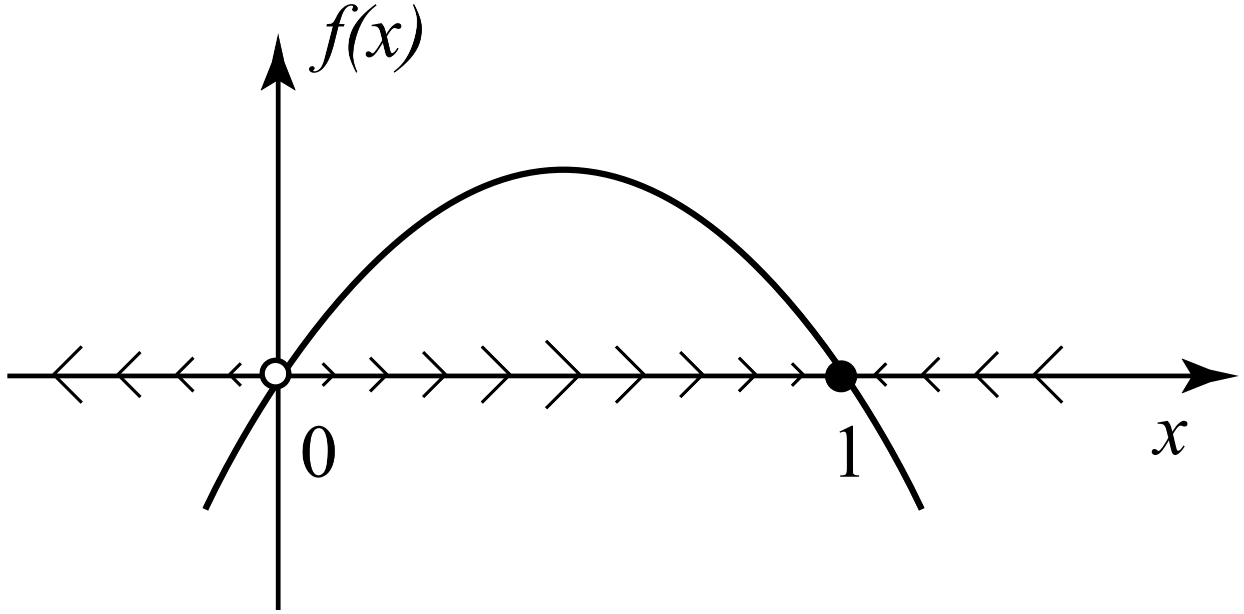

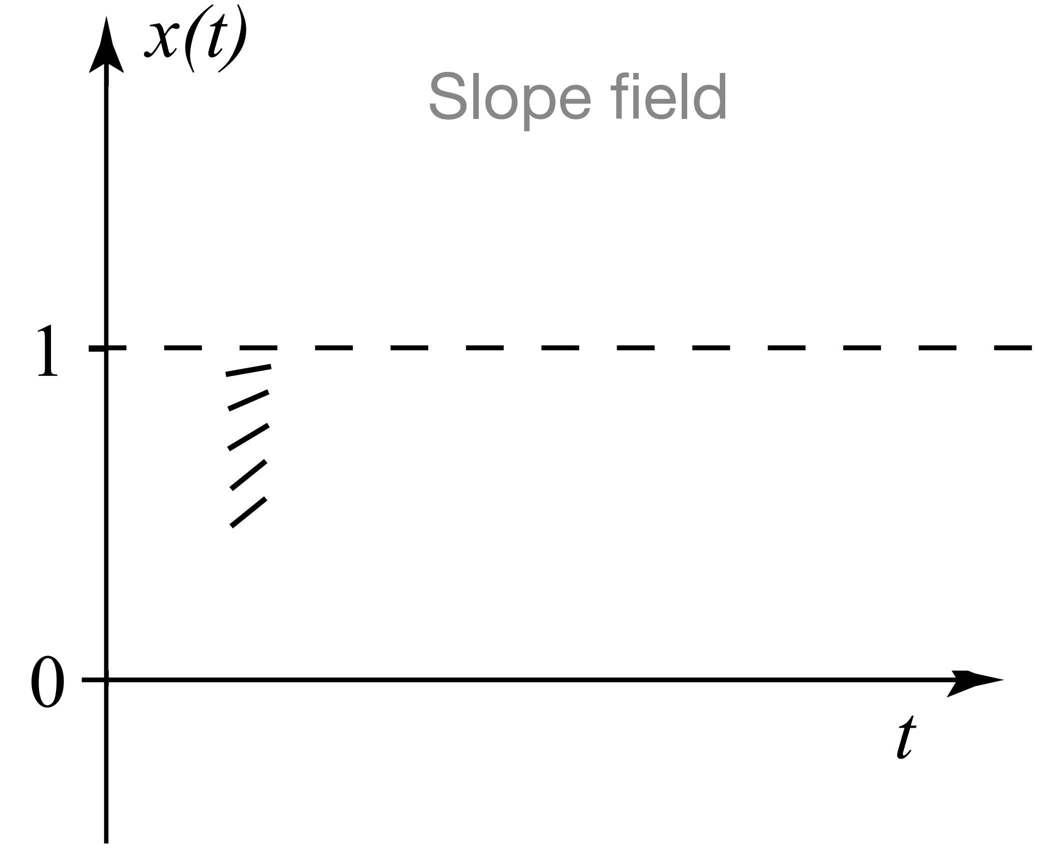

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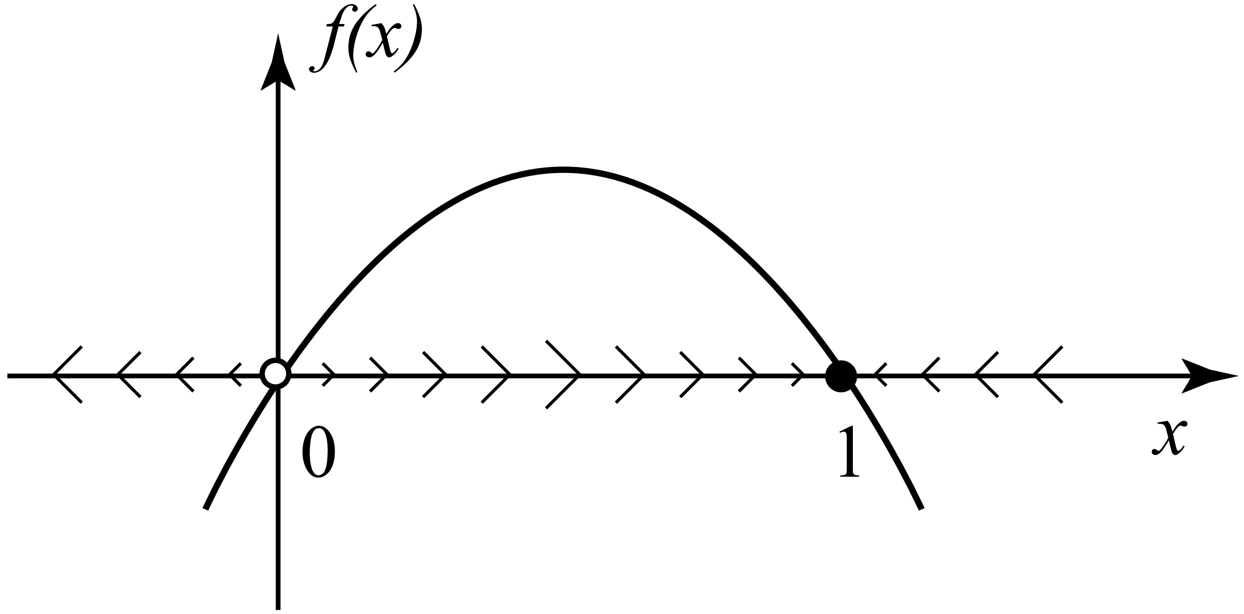

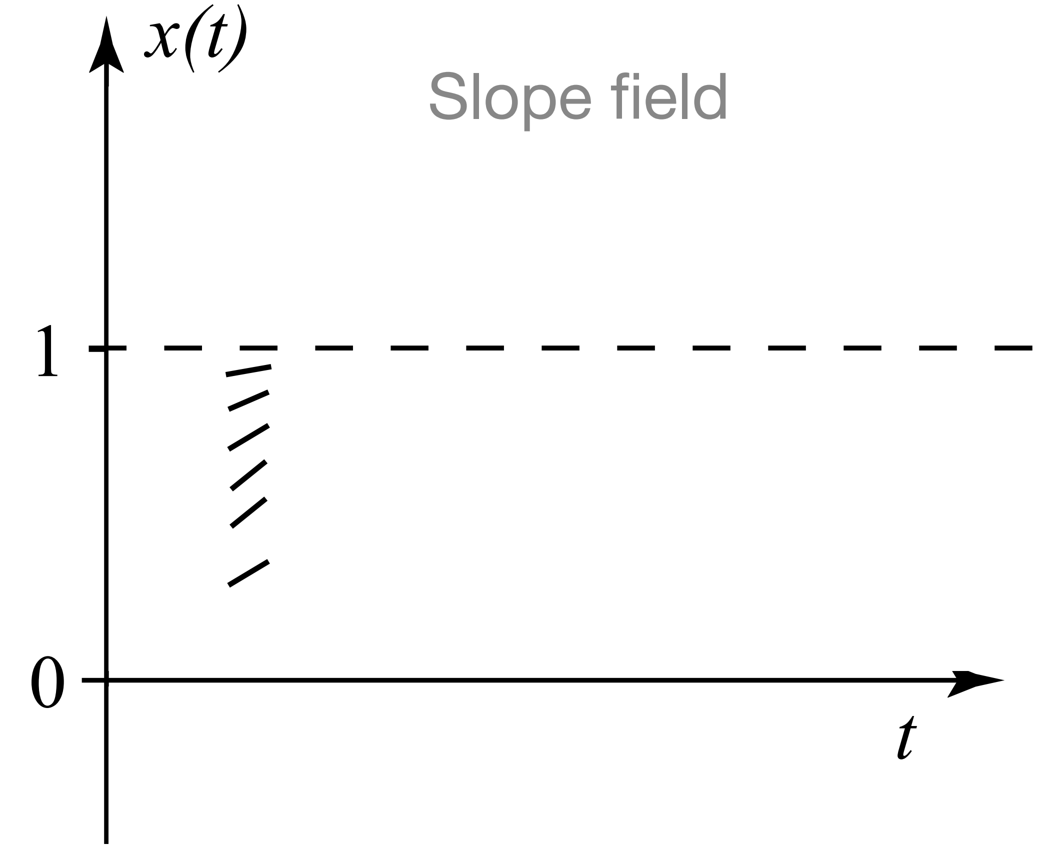

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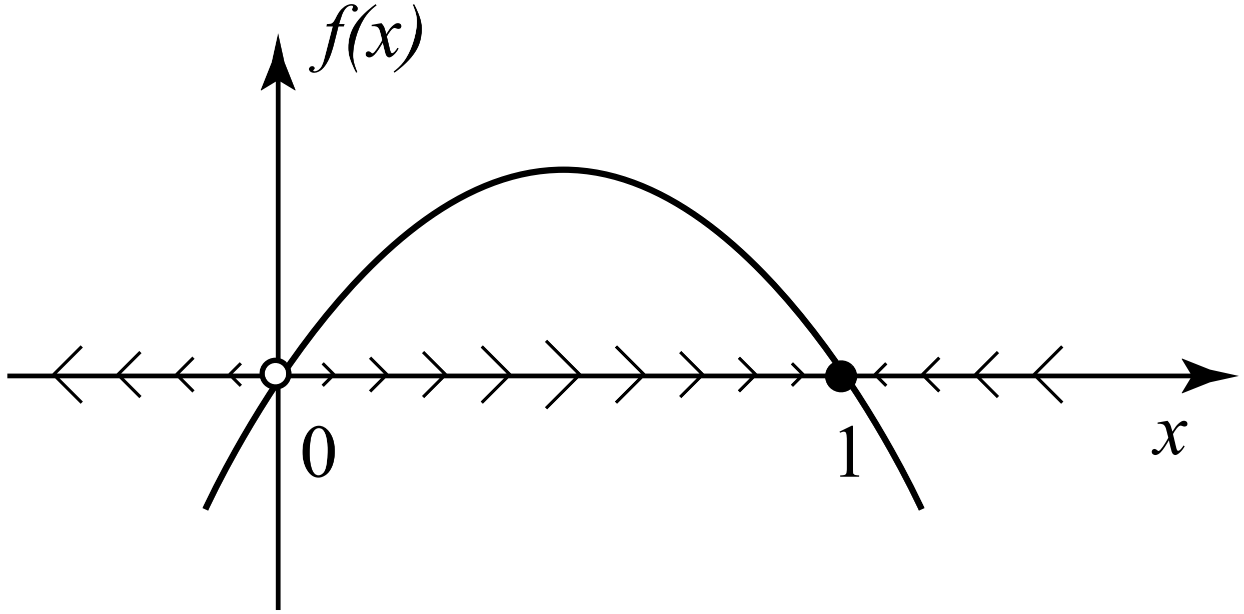

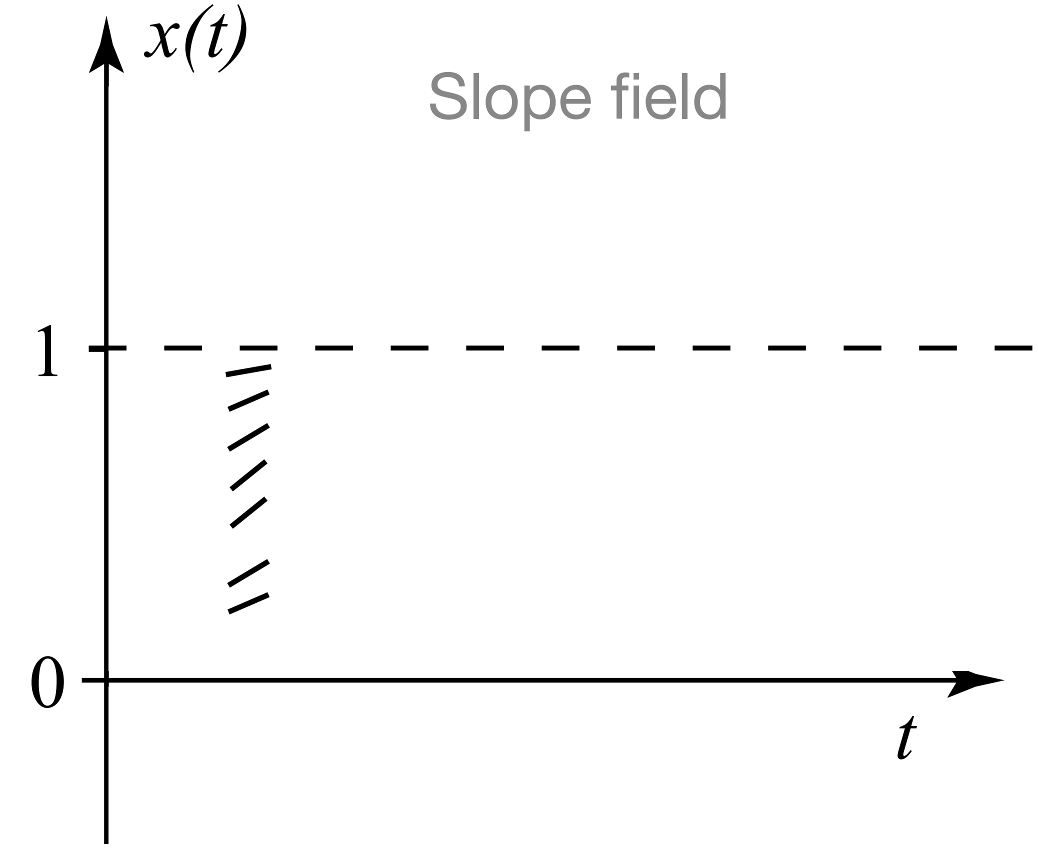

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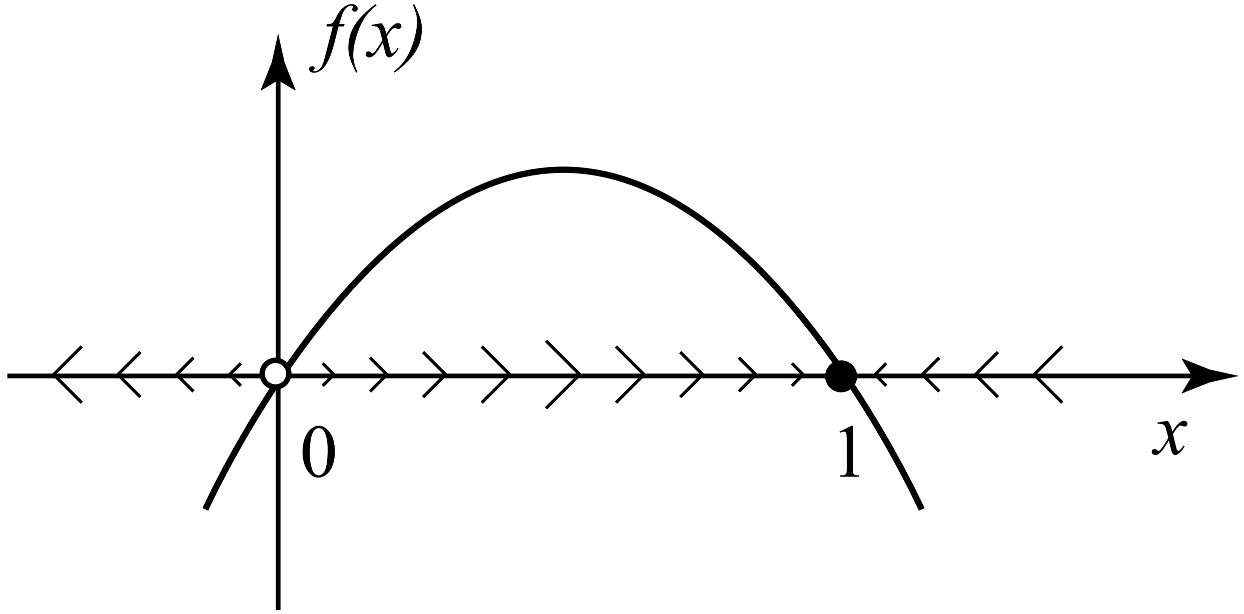

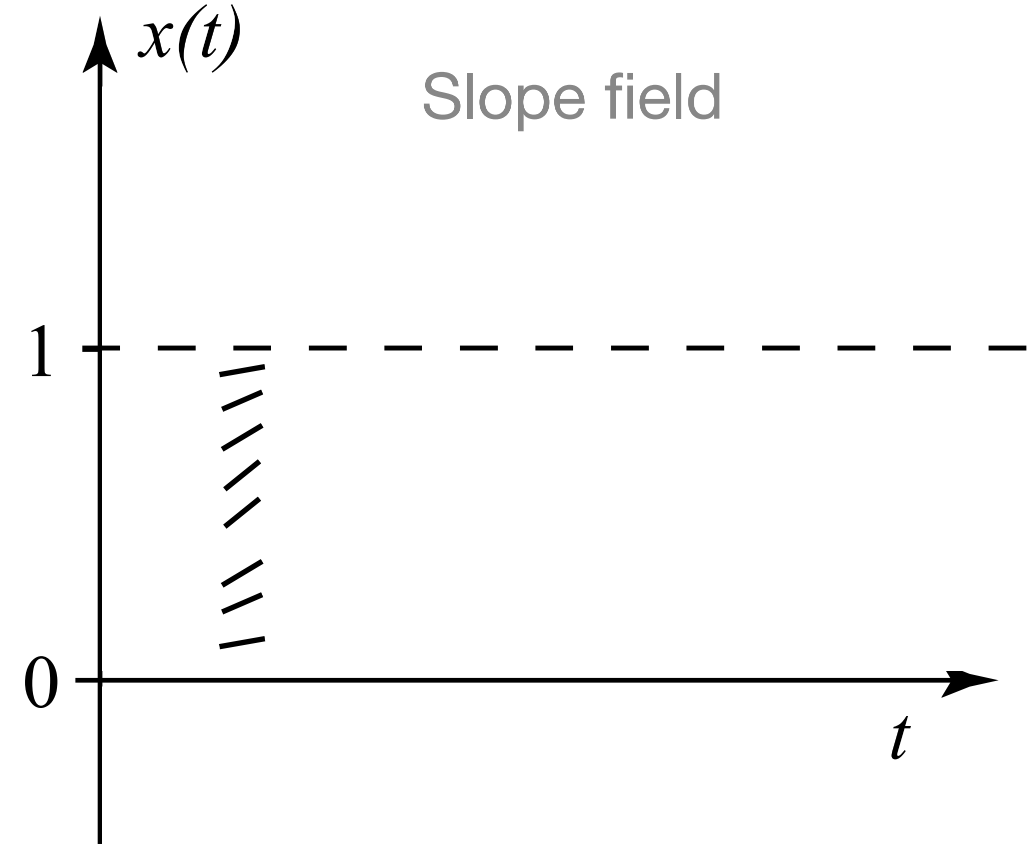

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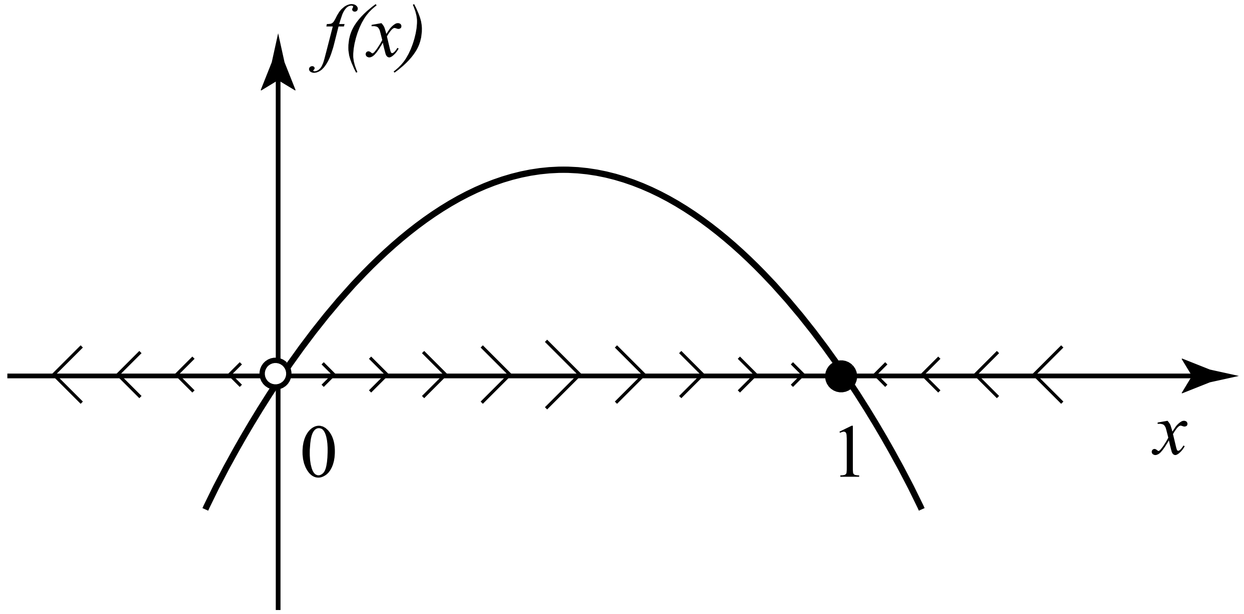

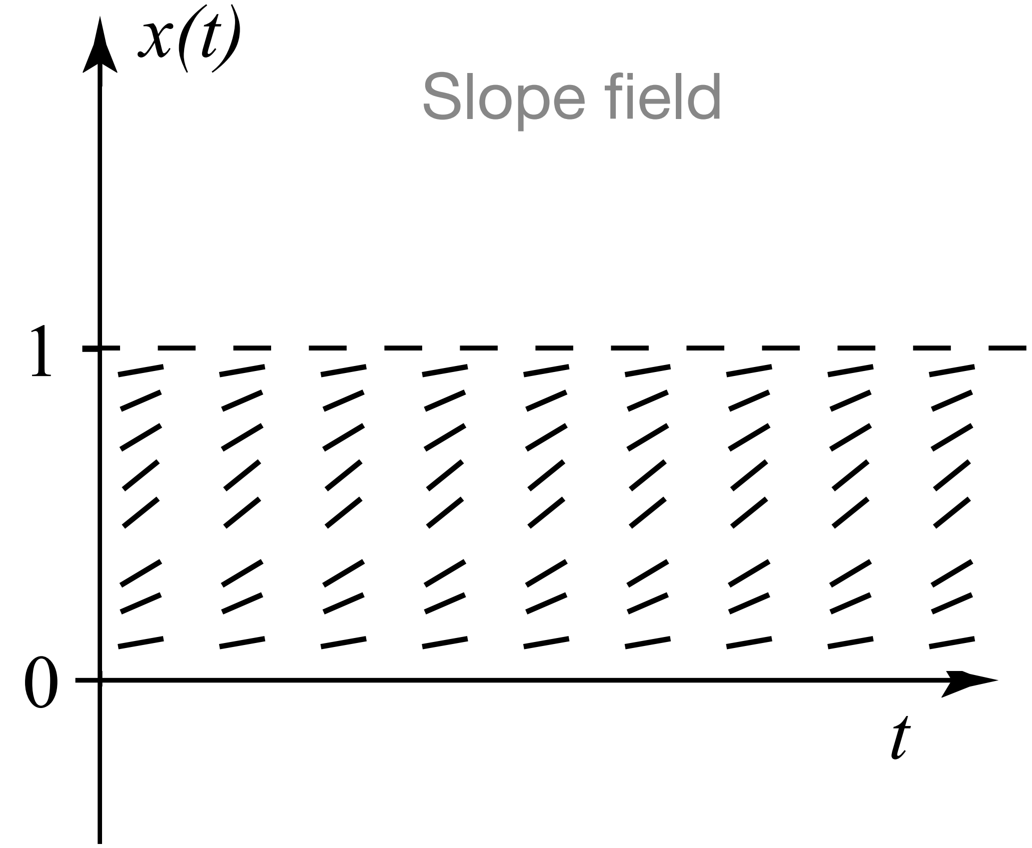

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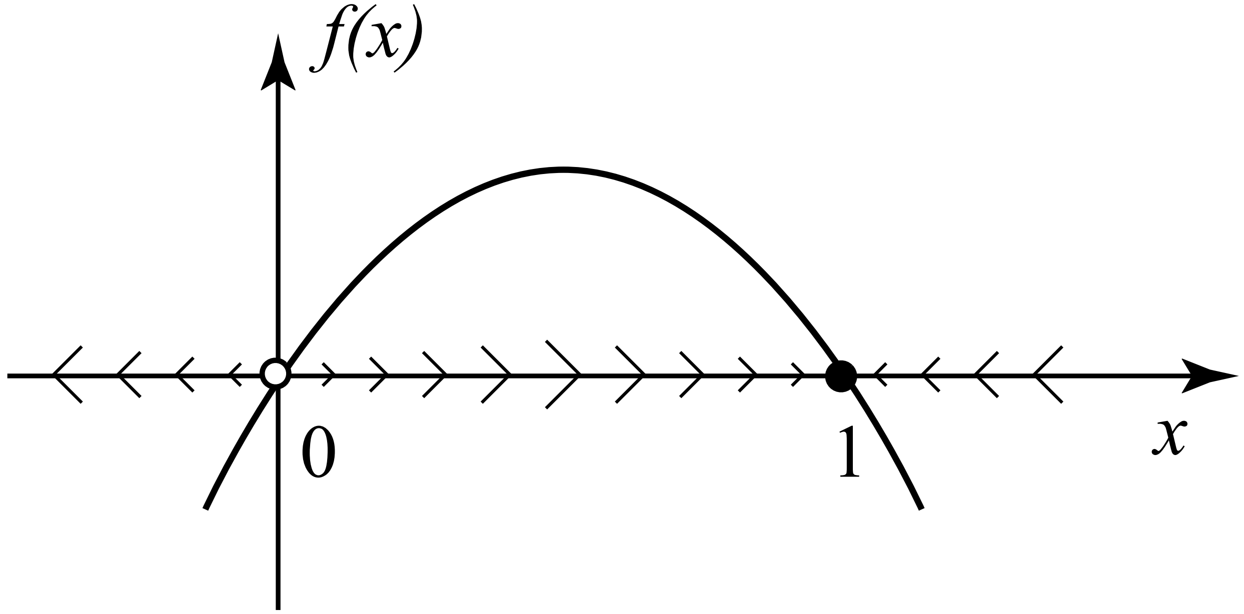

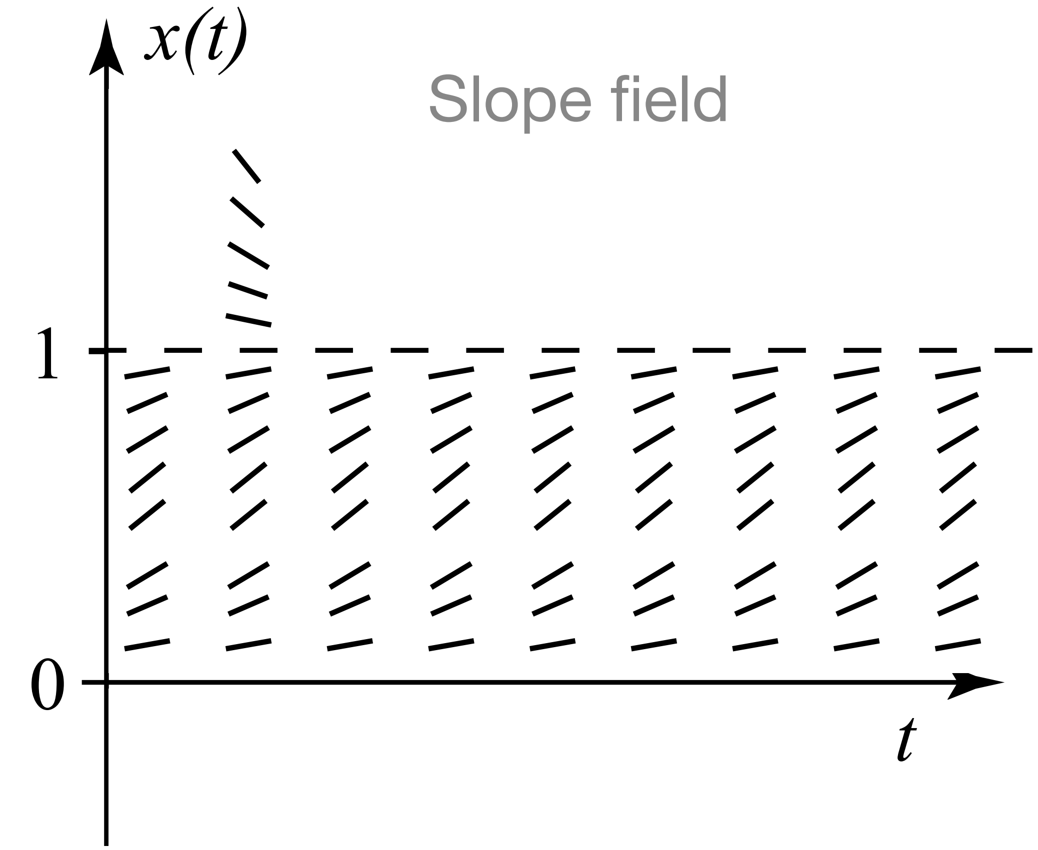

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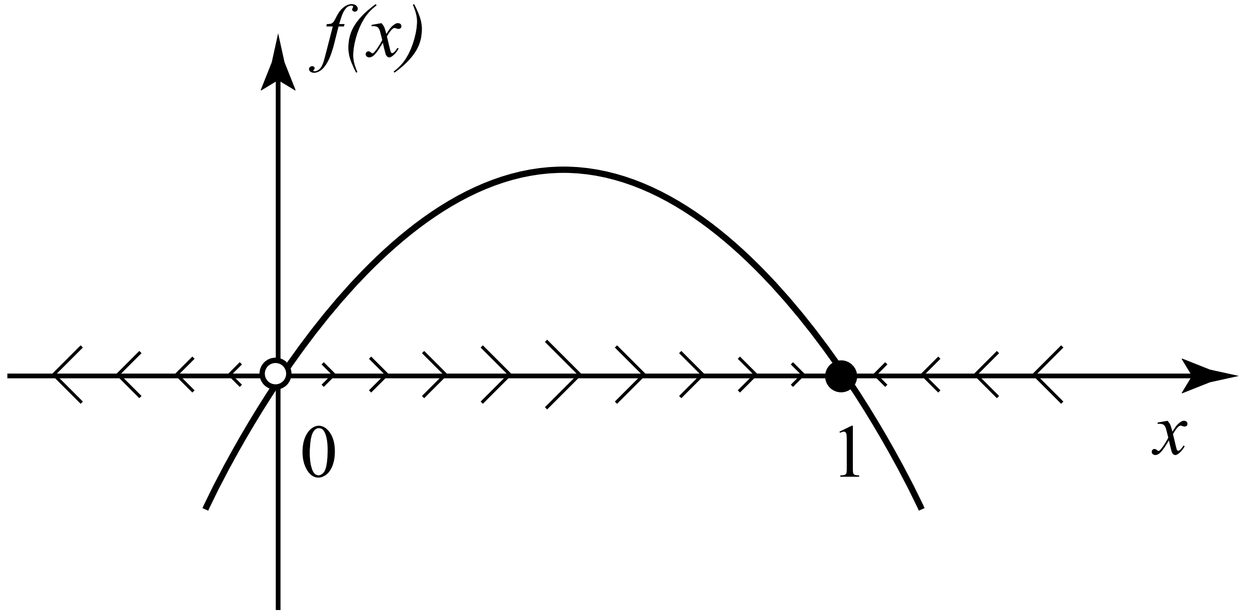

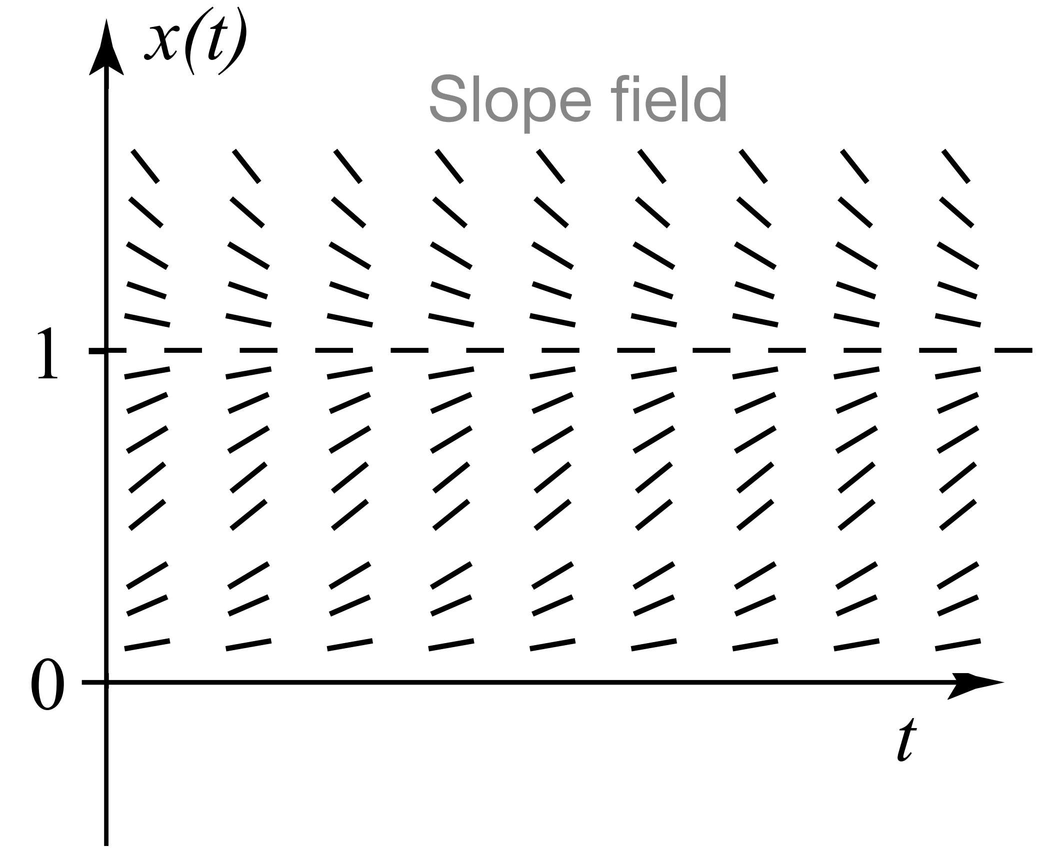

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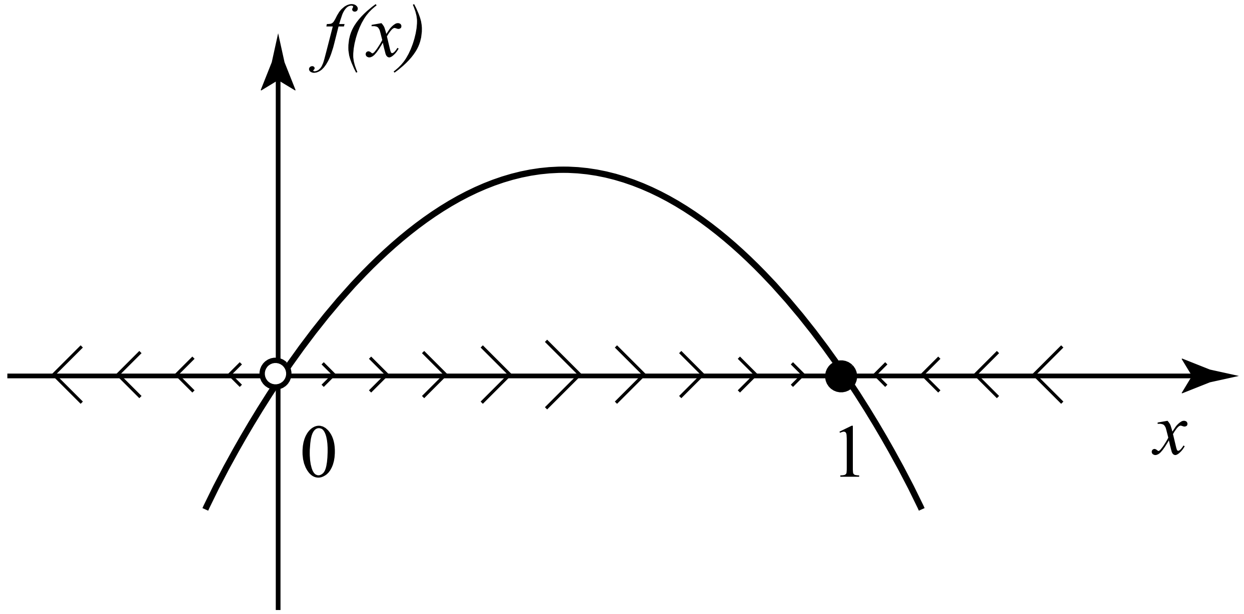

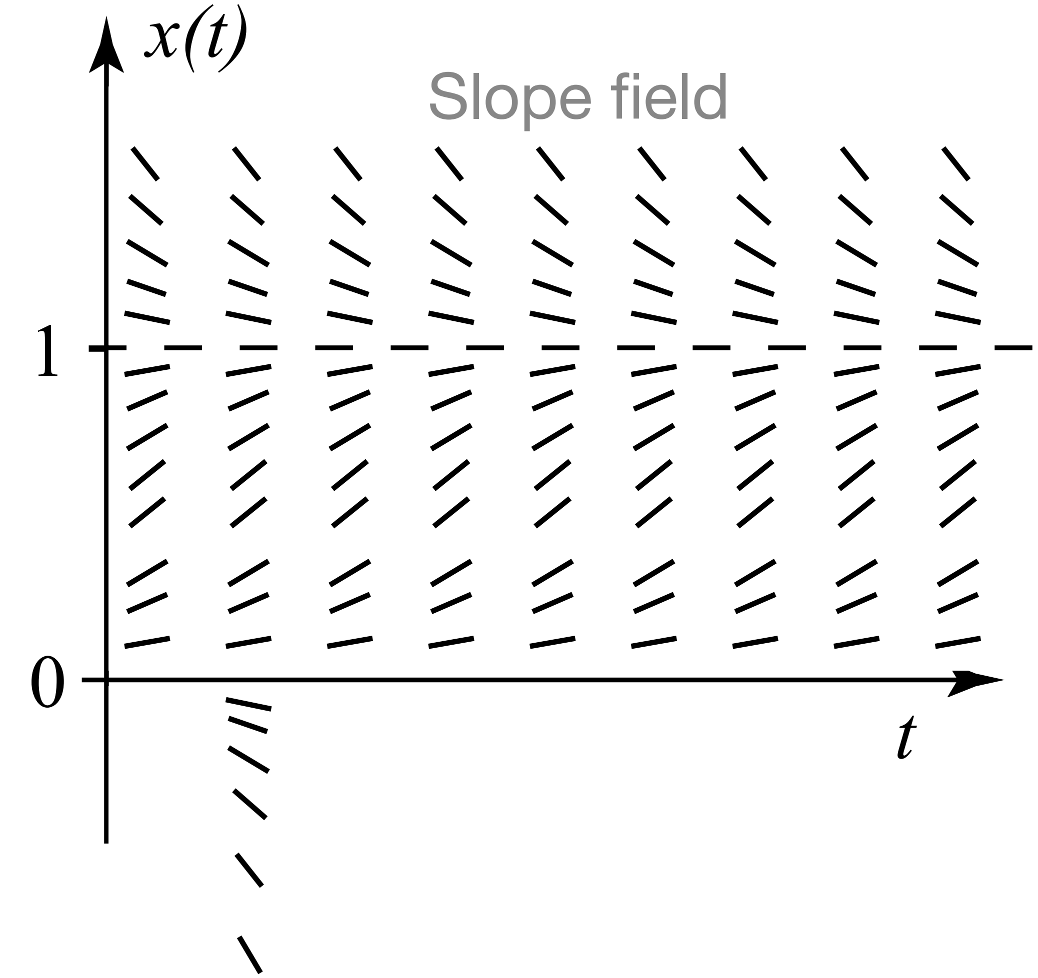

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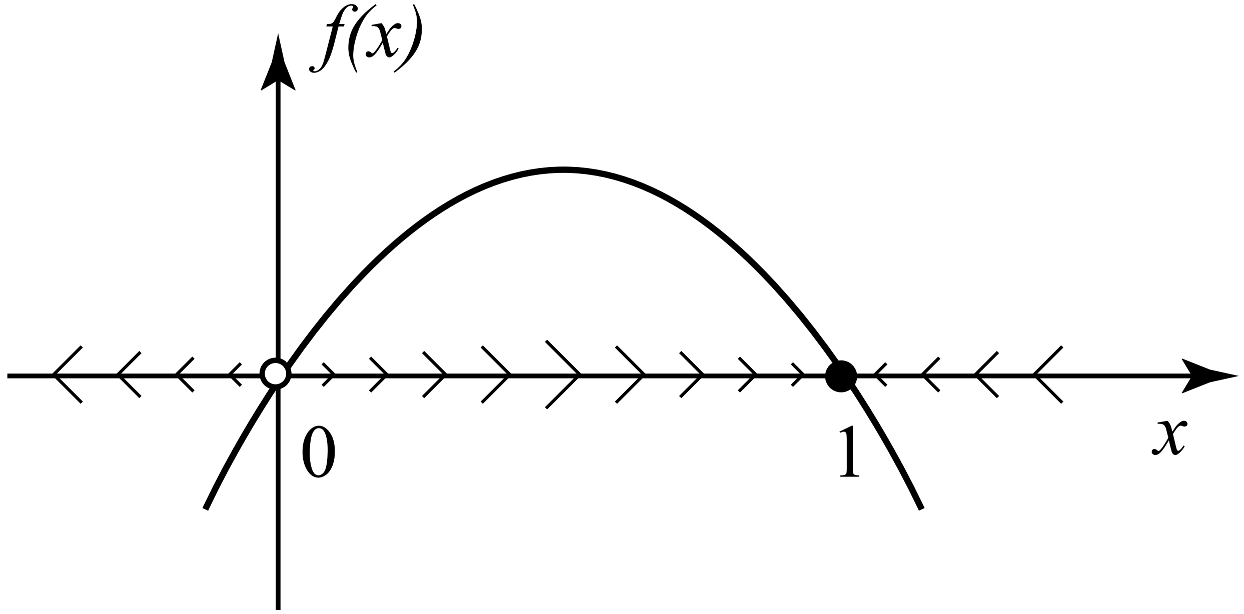

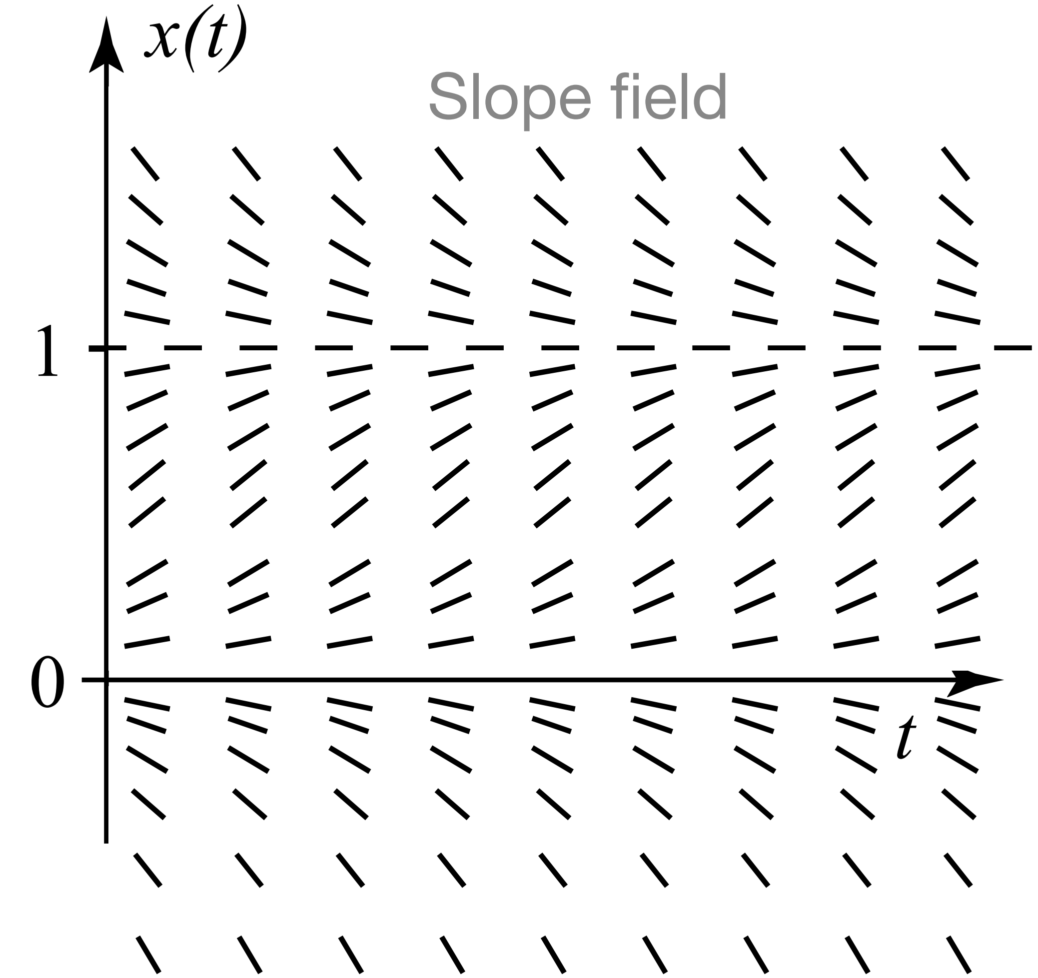

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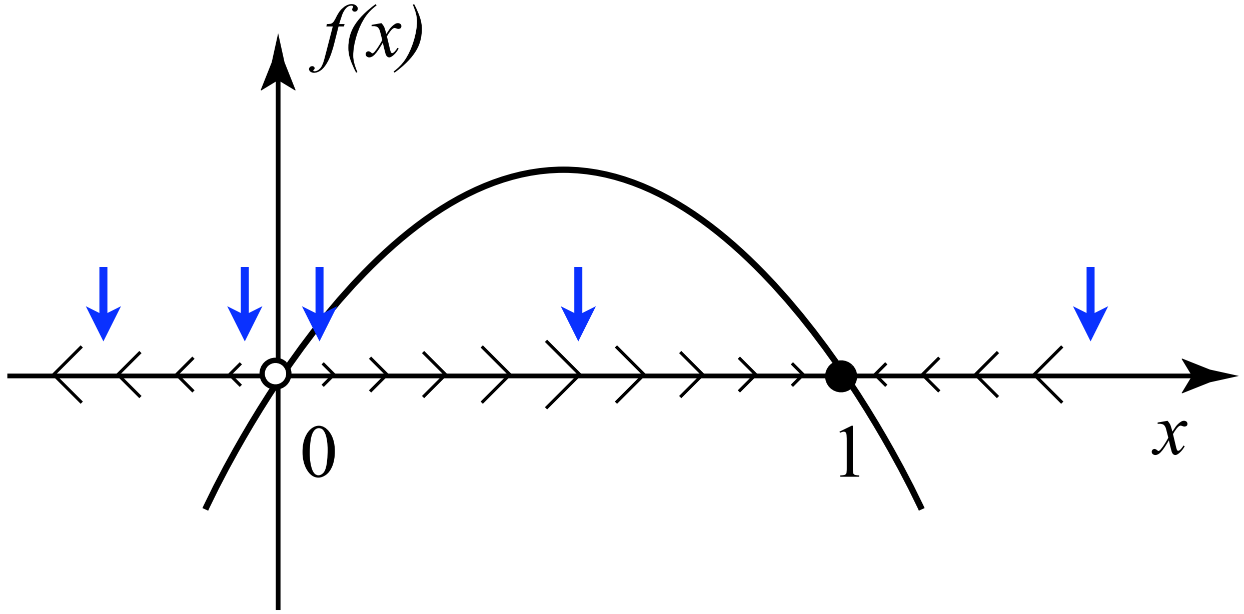

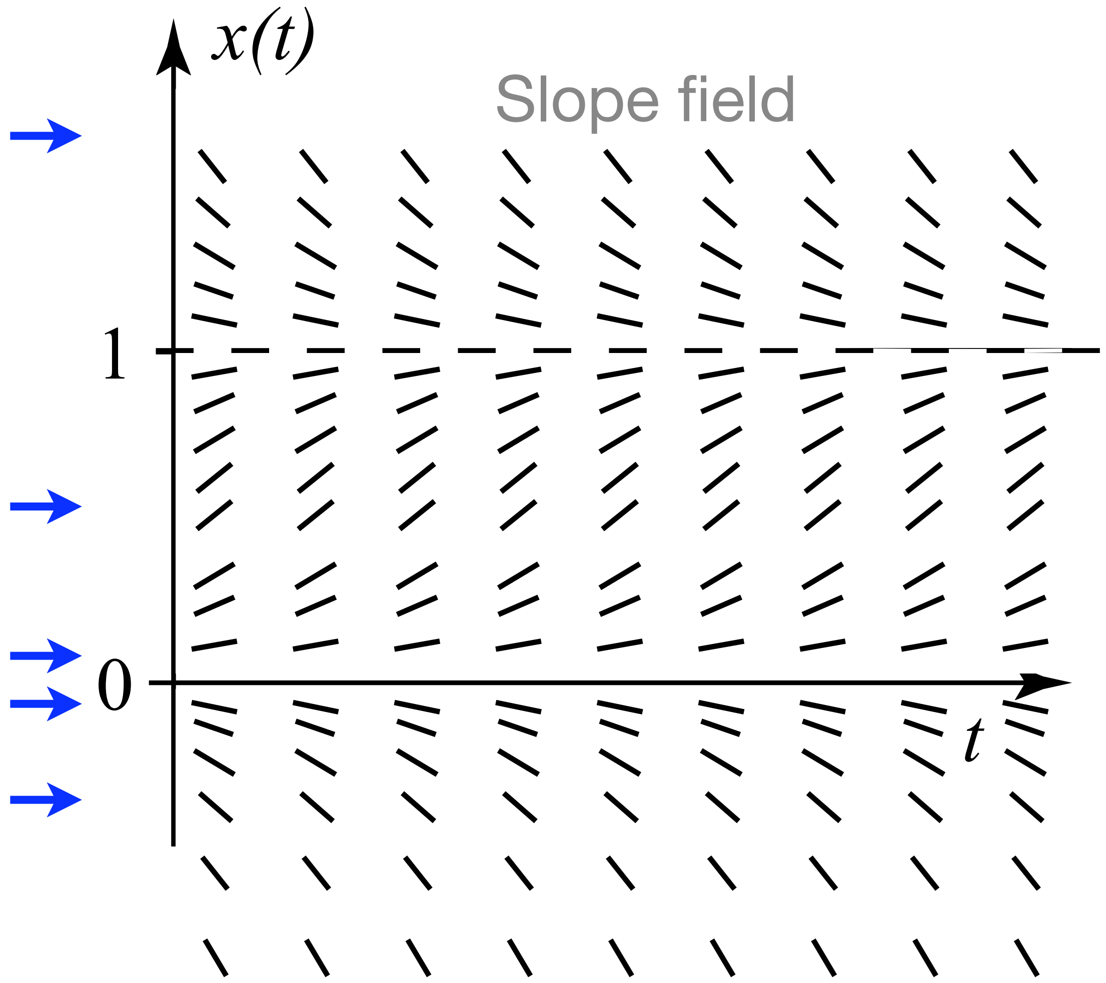

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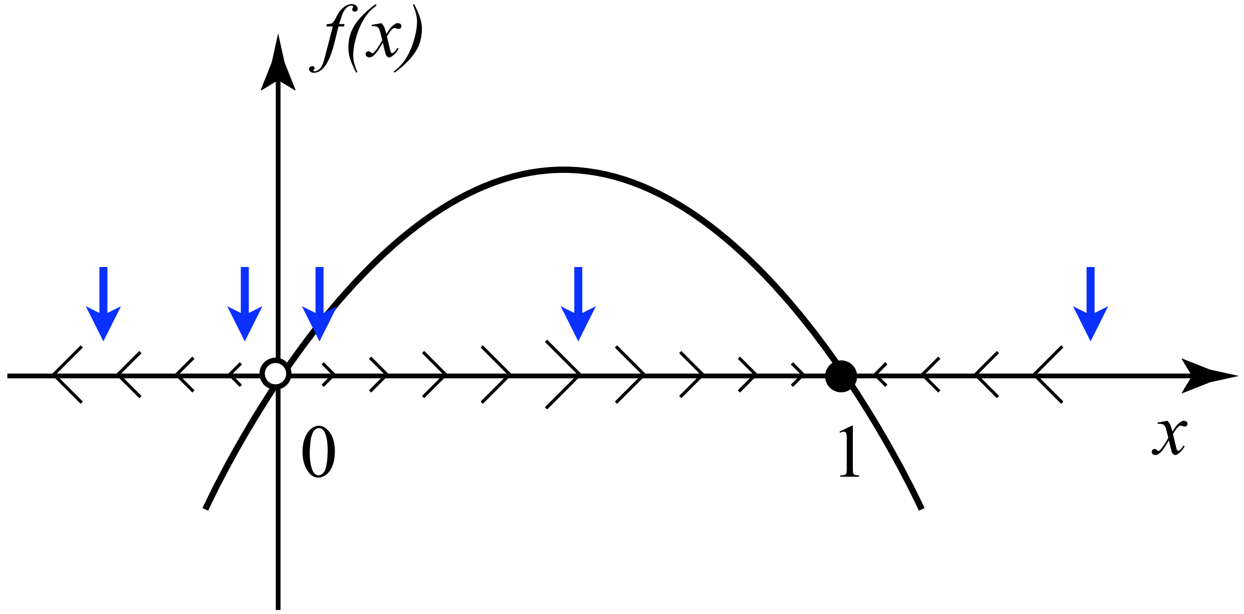

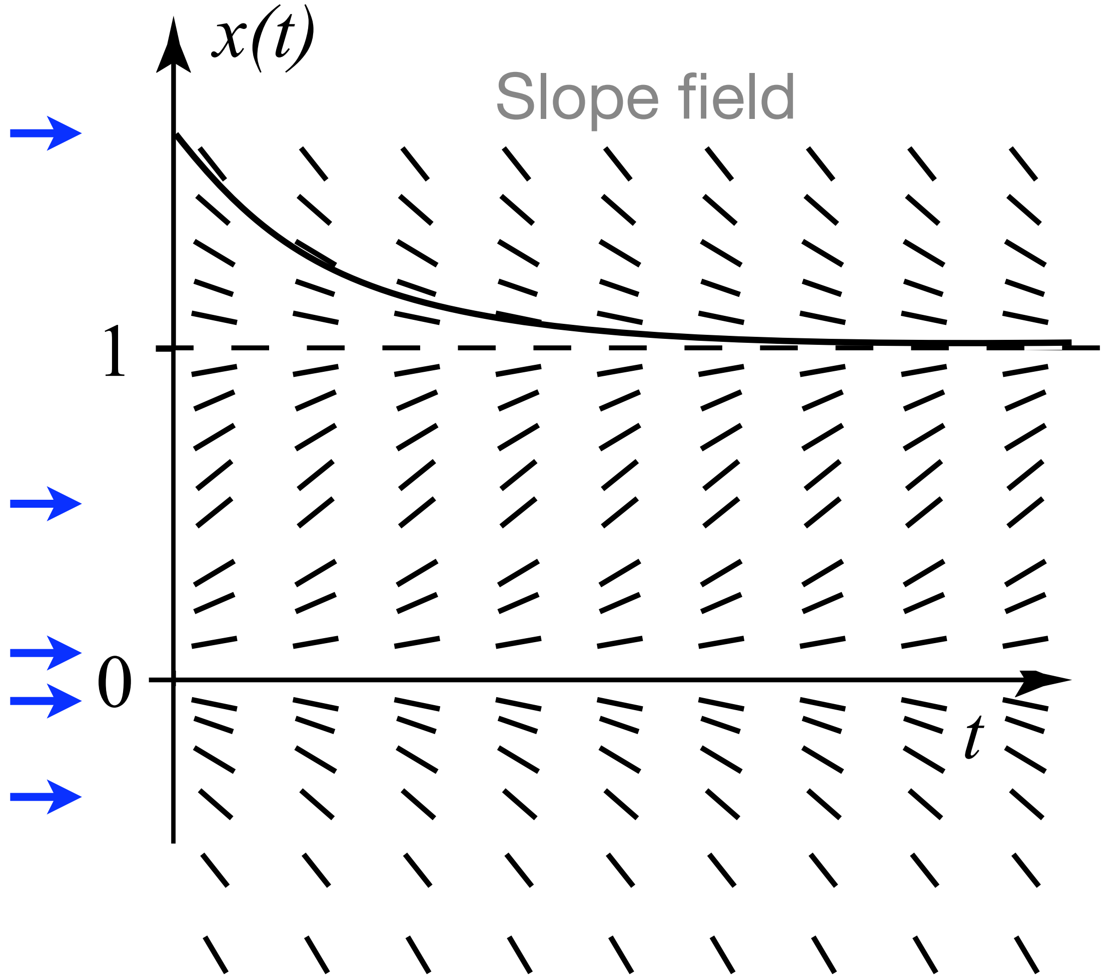

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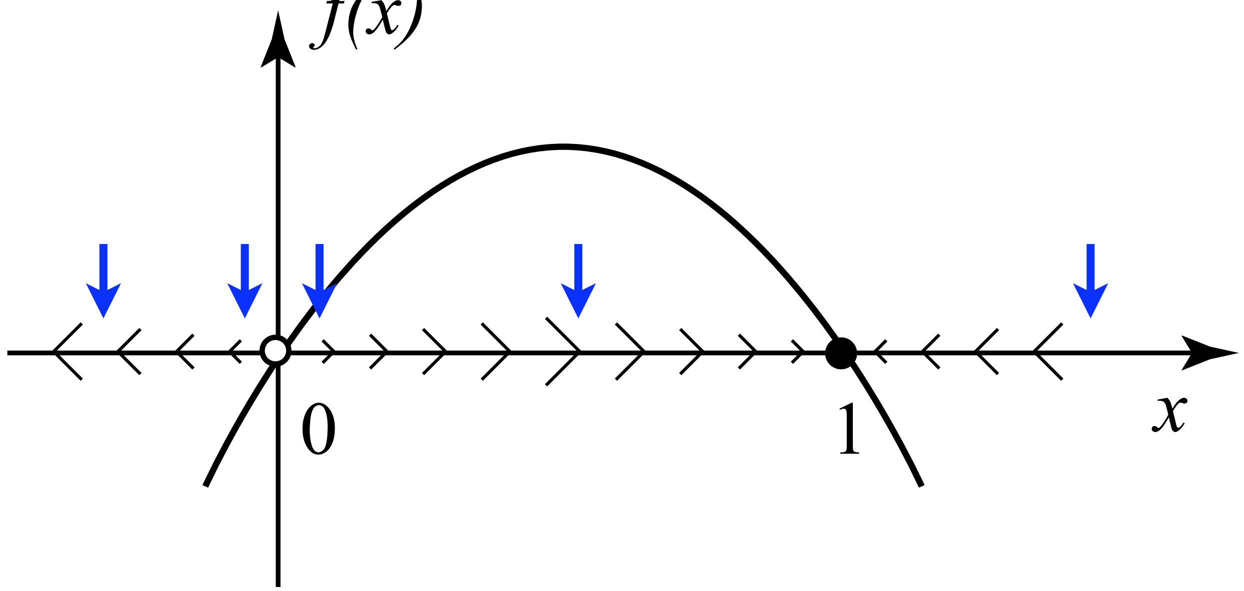

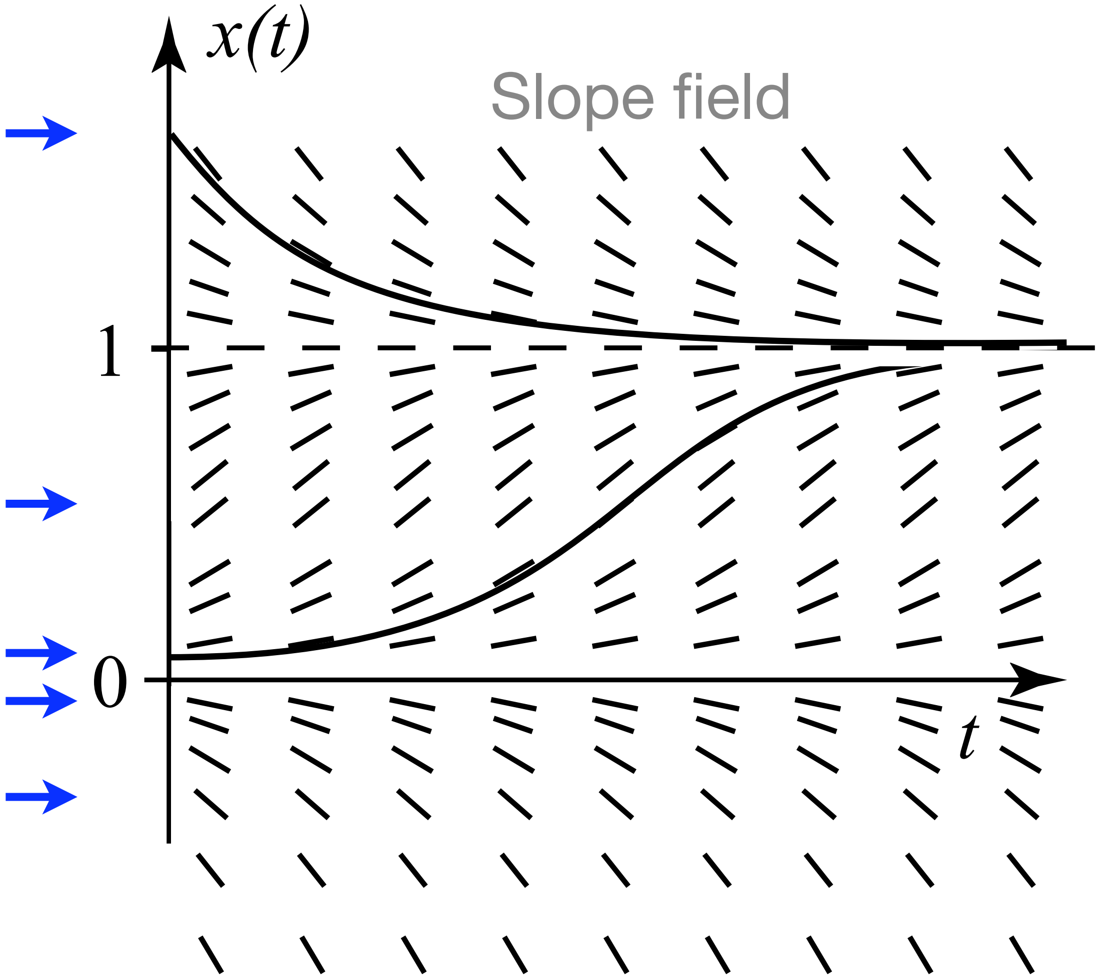

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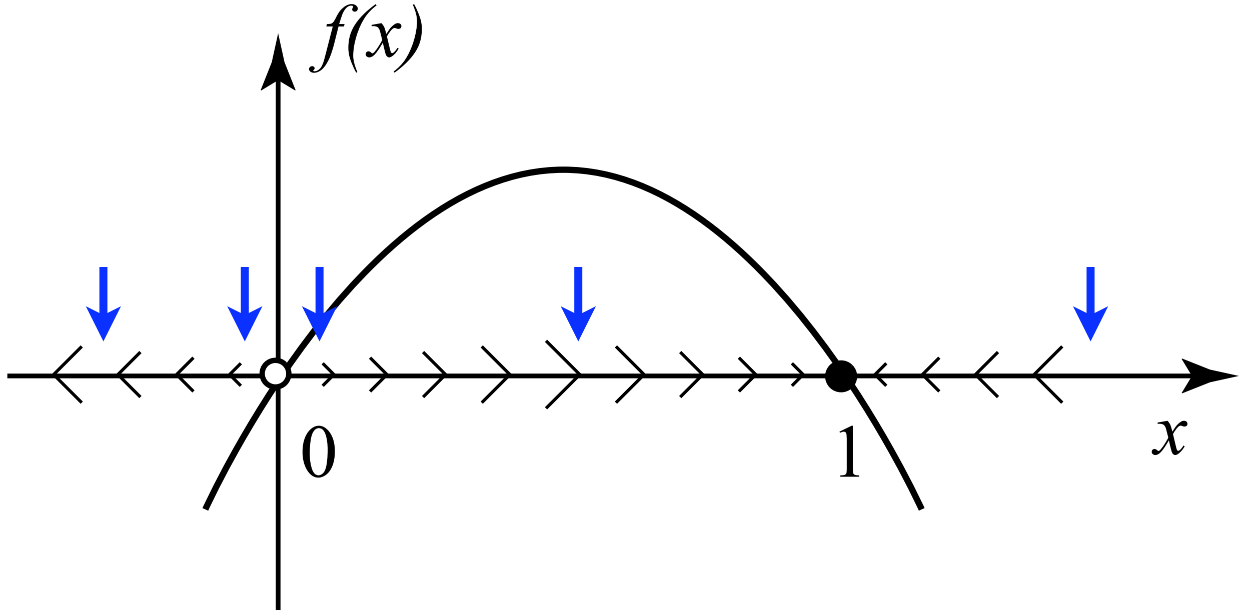

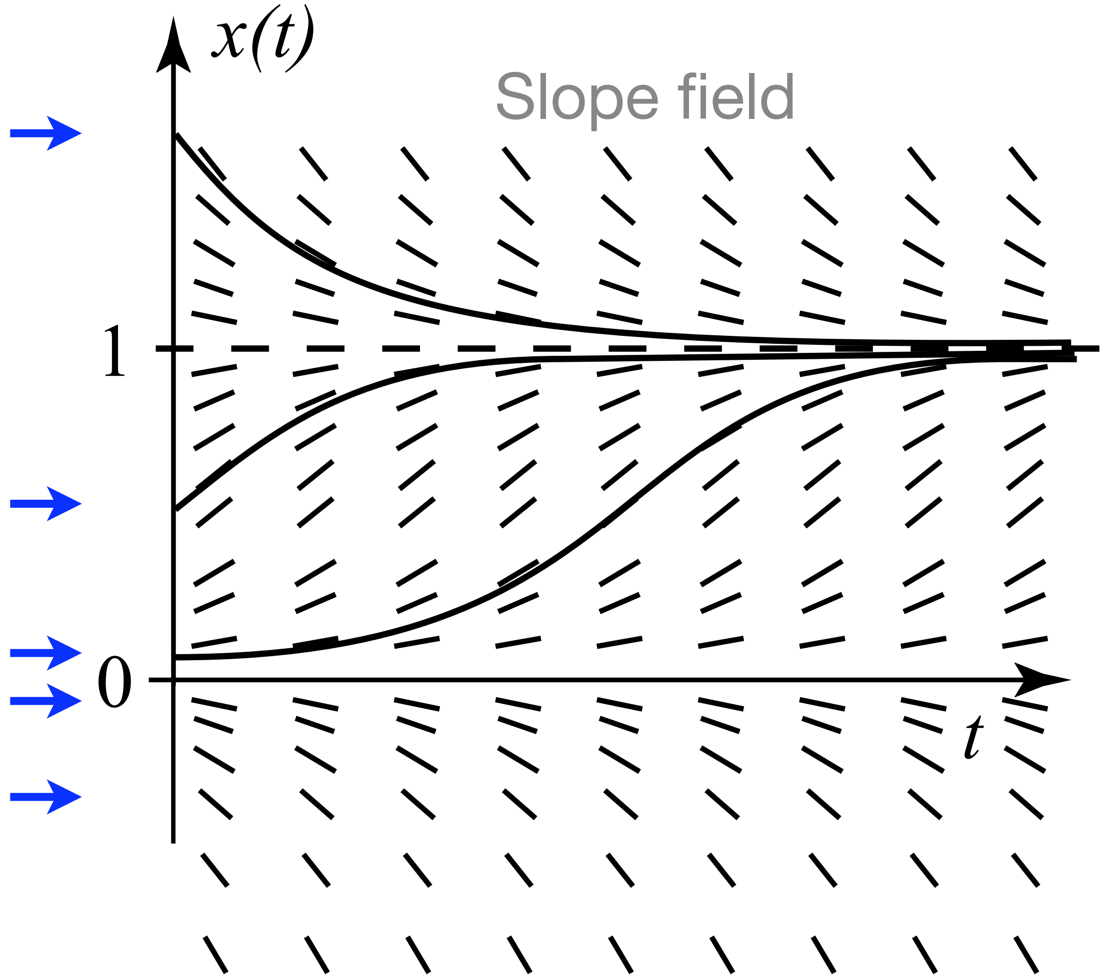

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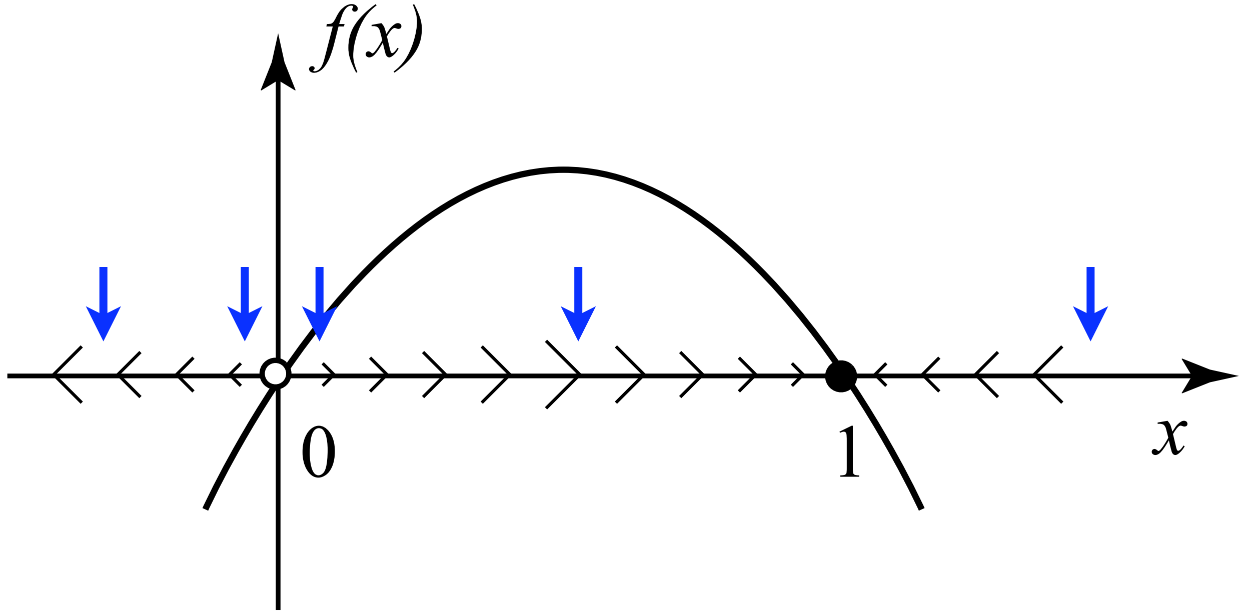

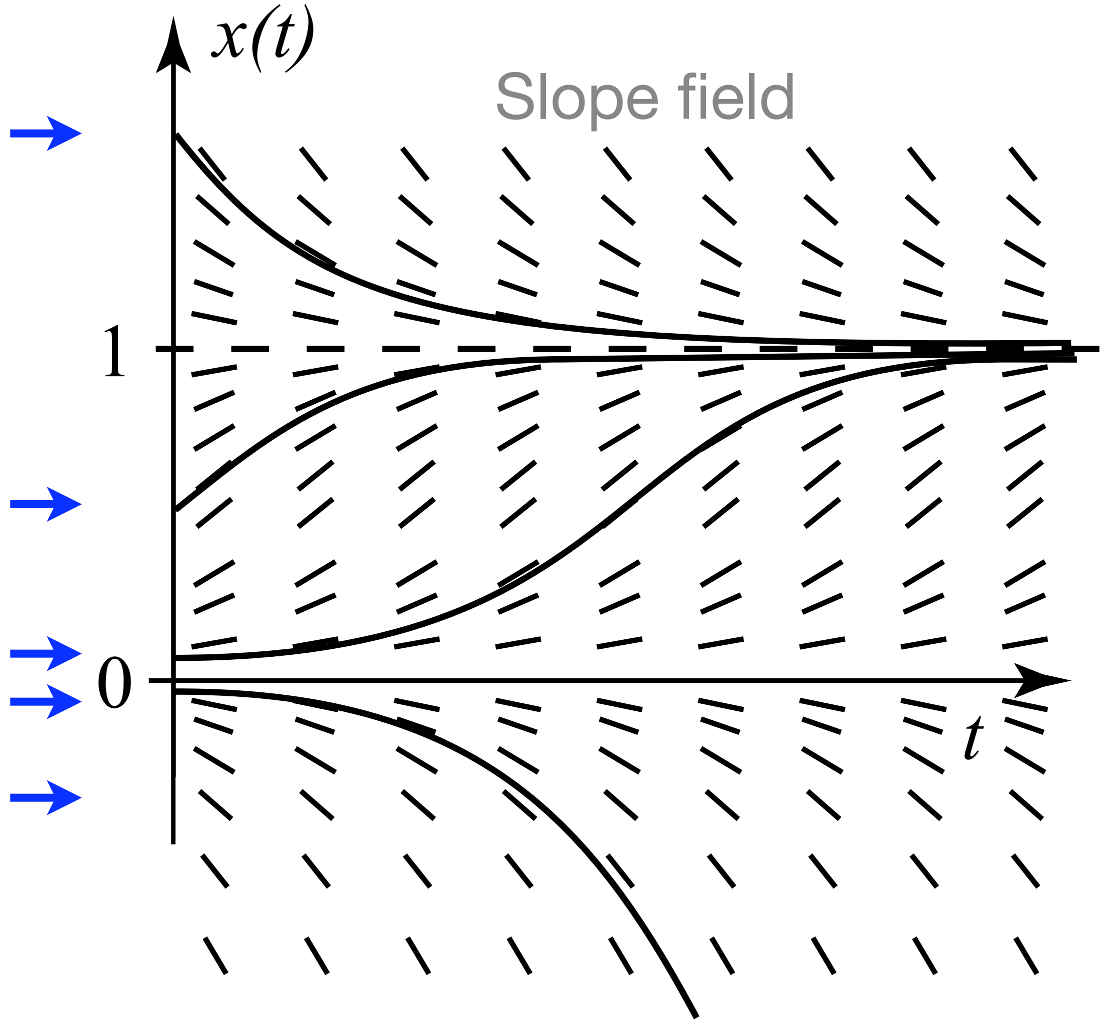

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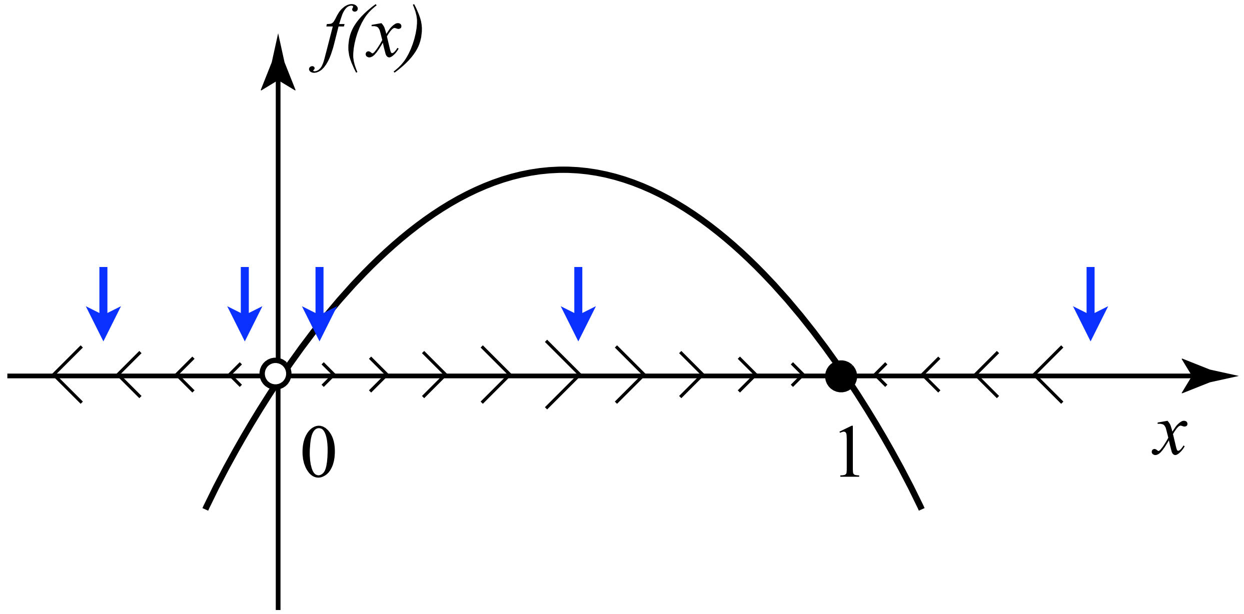

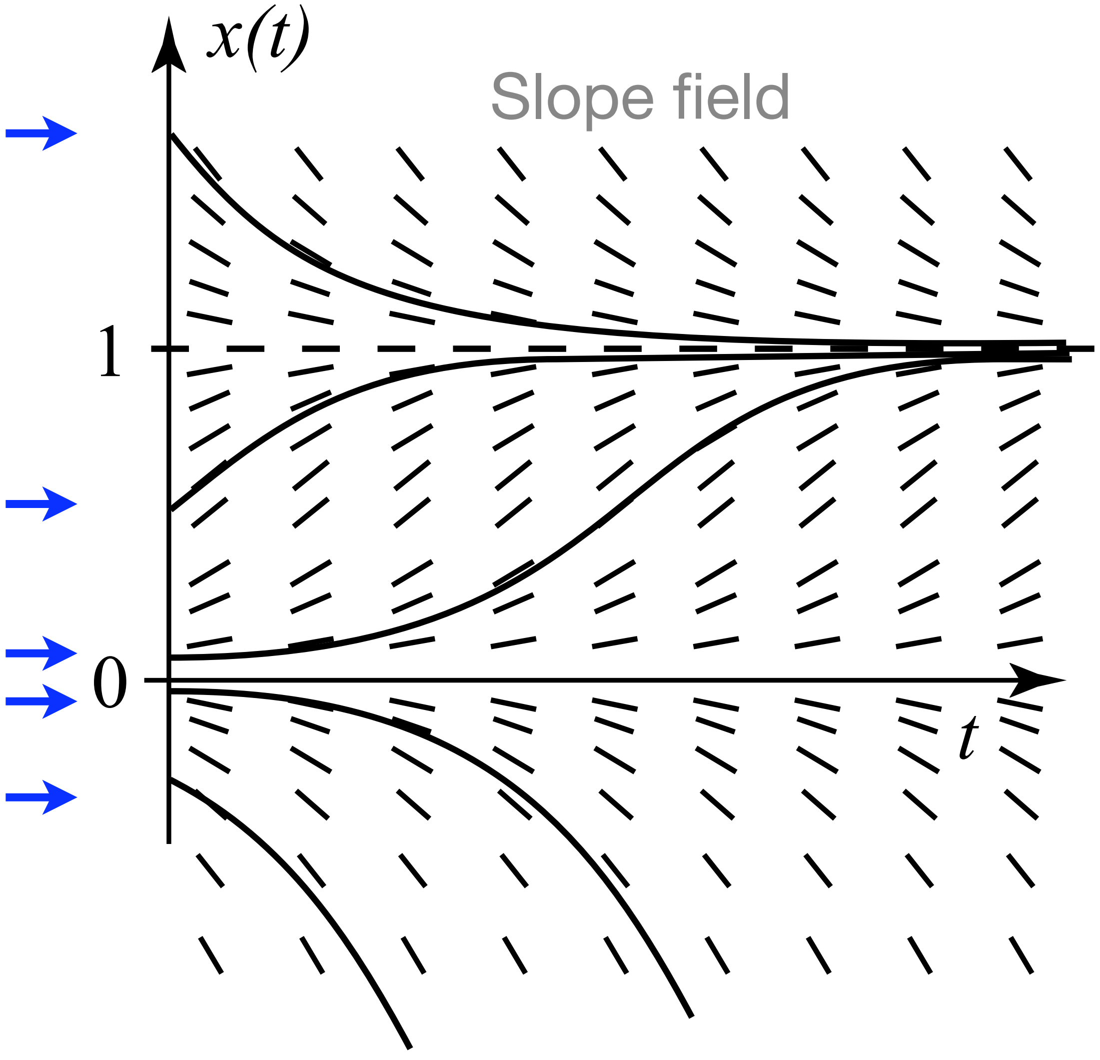

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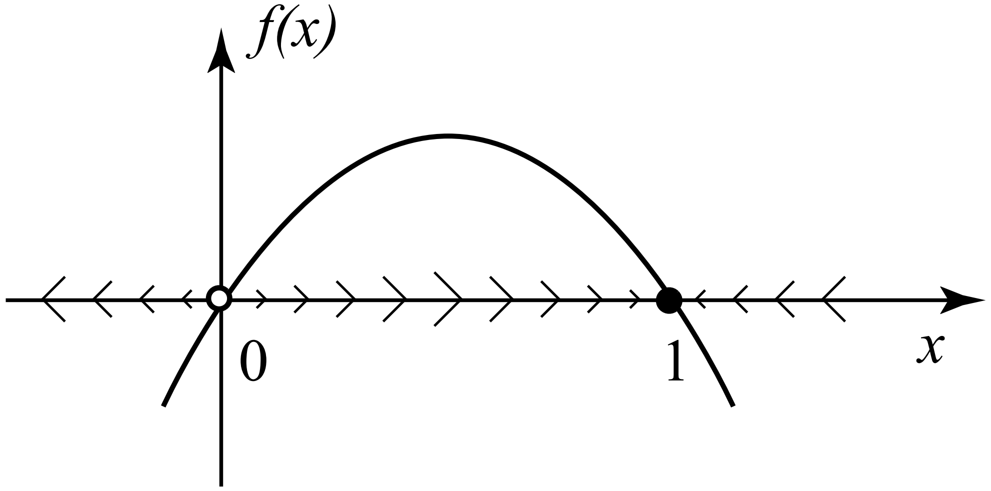

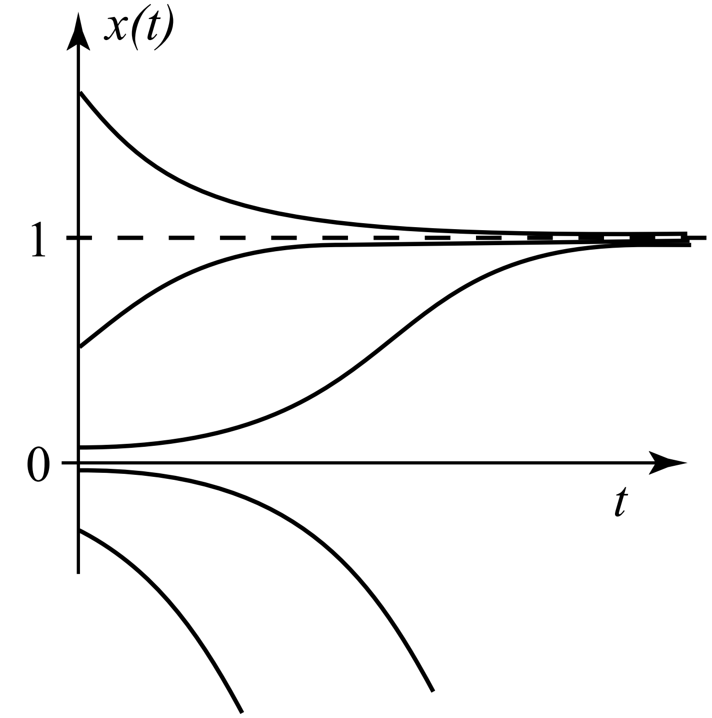

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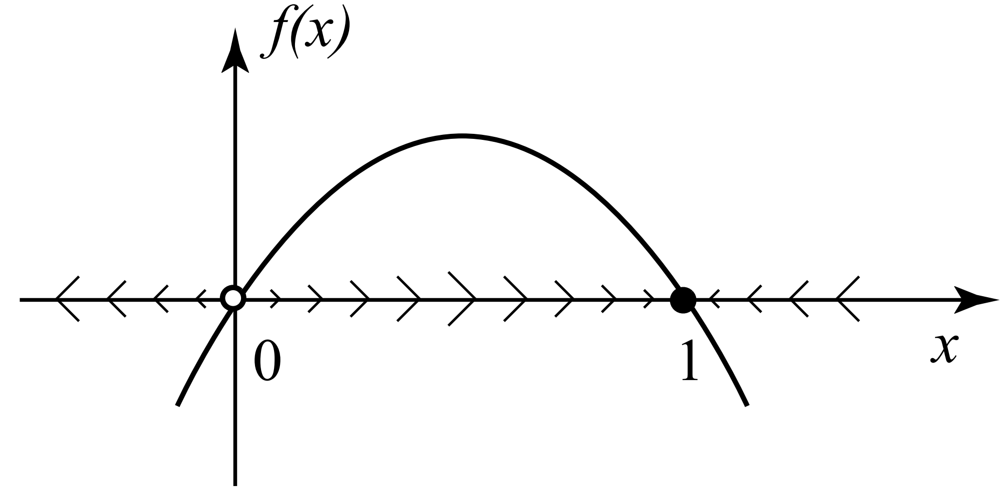

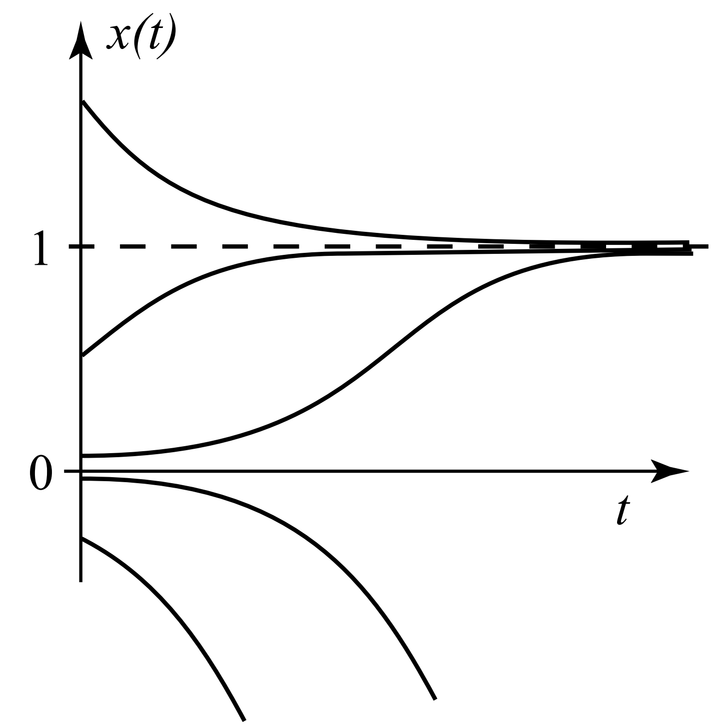

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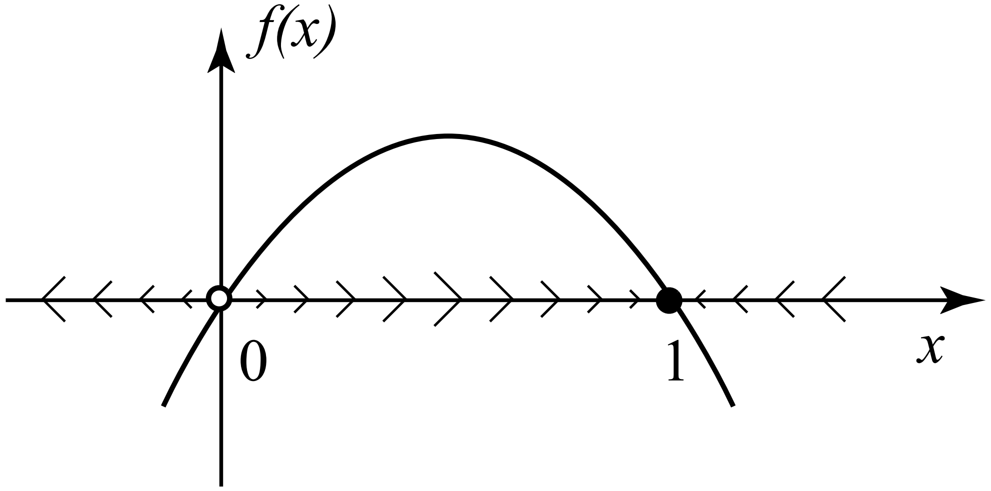
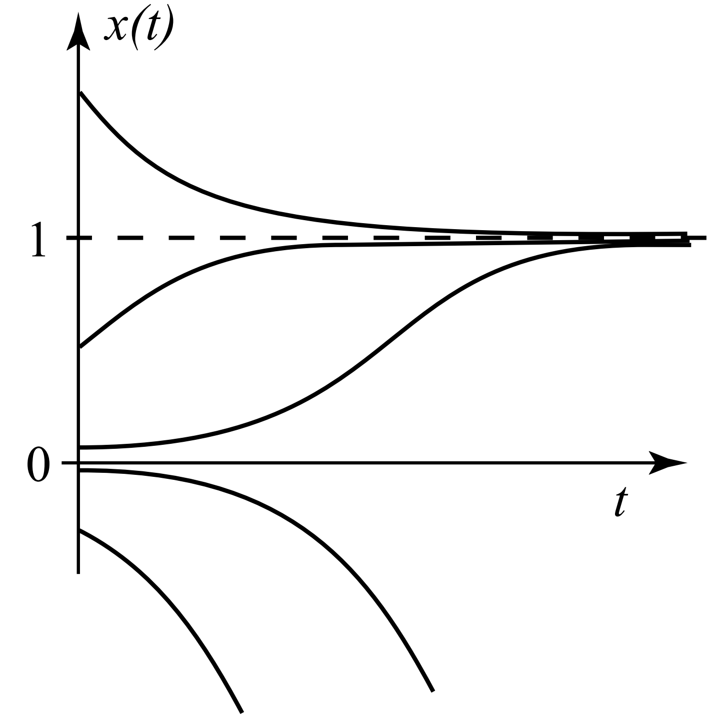
$$\frac{d^2x}{dt^2} =$$





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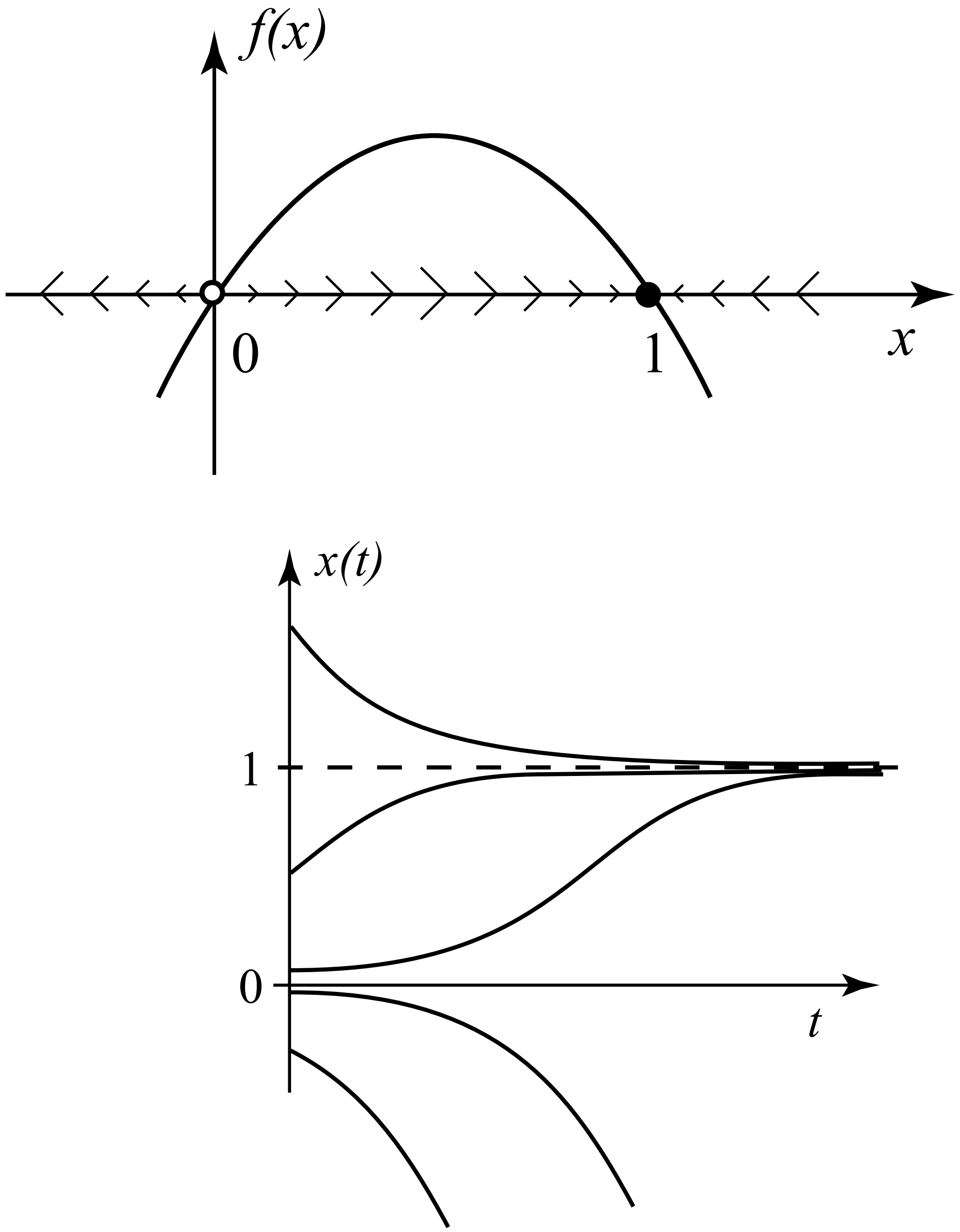

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$$\frac{dx}{dt} = x - x^2 = f(x).$$

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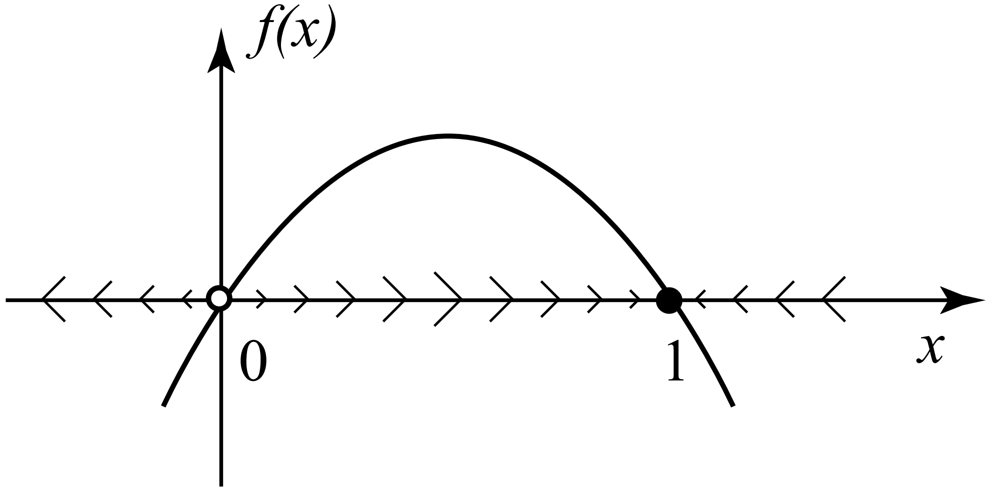

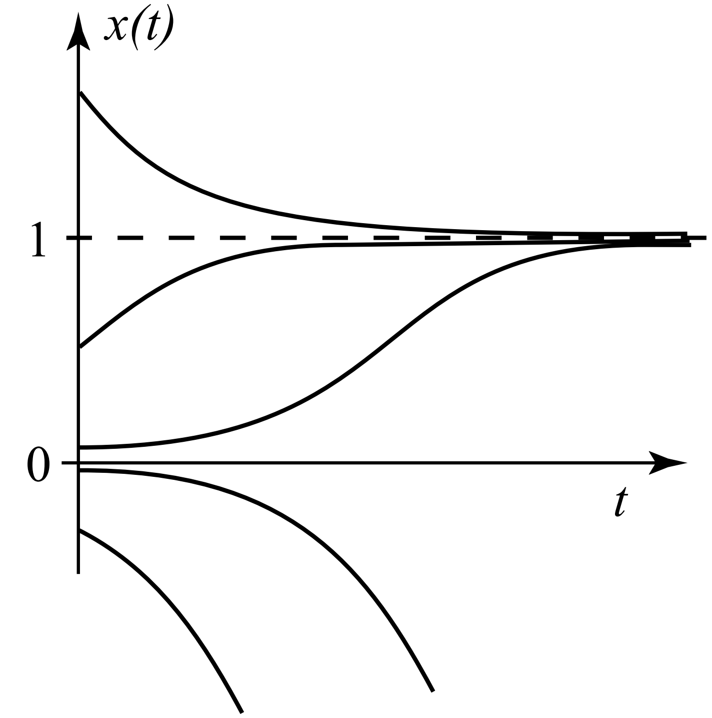

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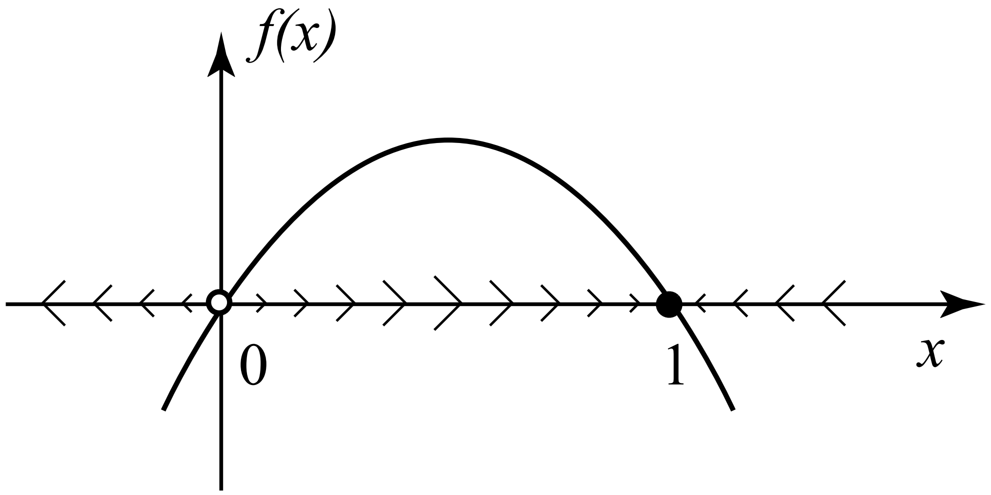
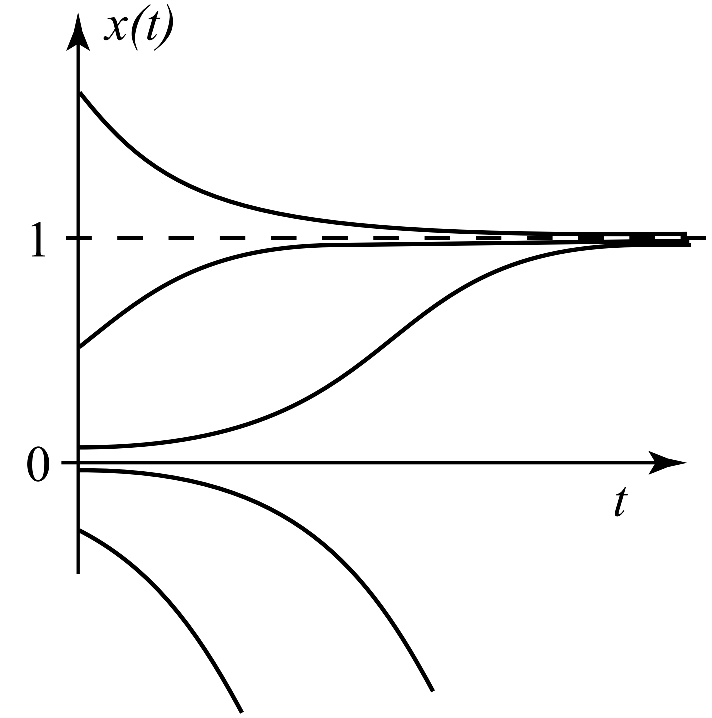
- Either $\frac{dx}{dt} = 0$ or $f'(x(t)) = 0$.





Phase line

- Where do inflection points occur?

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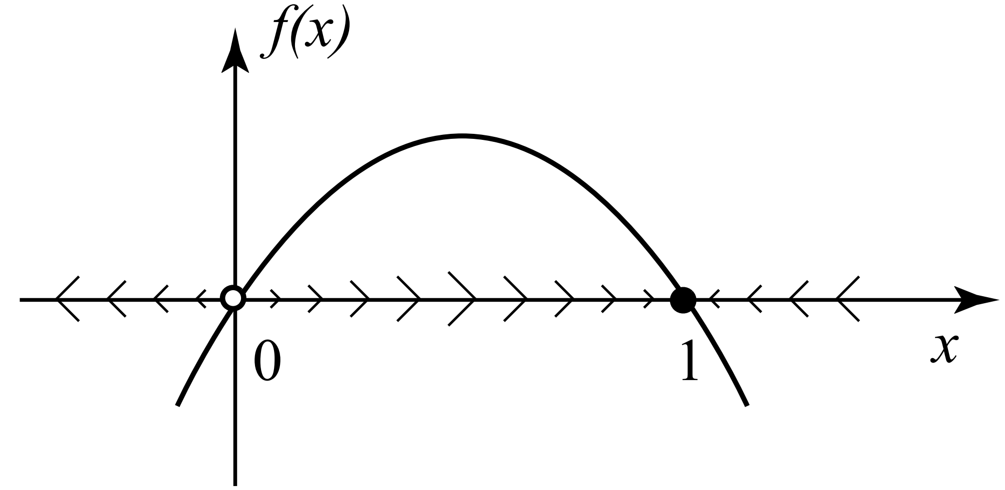

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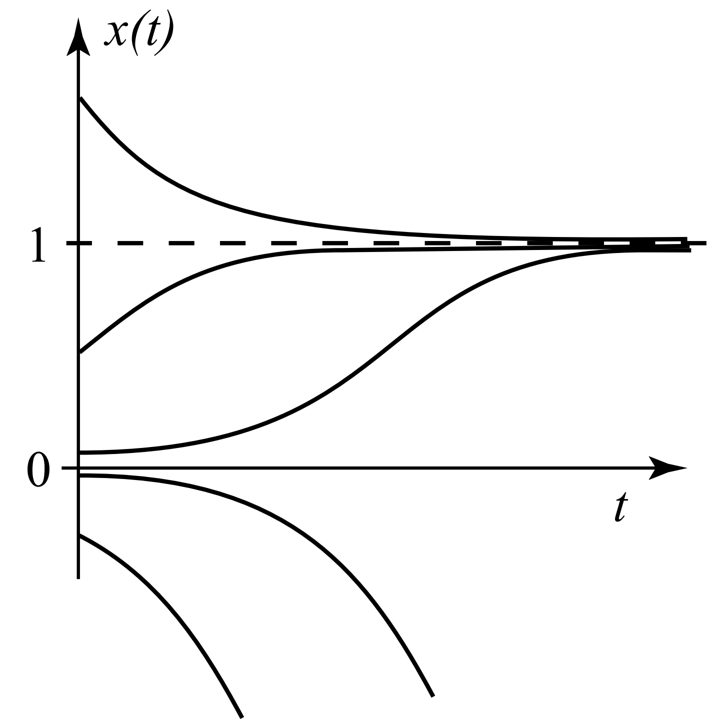

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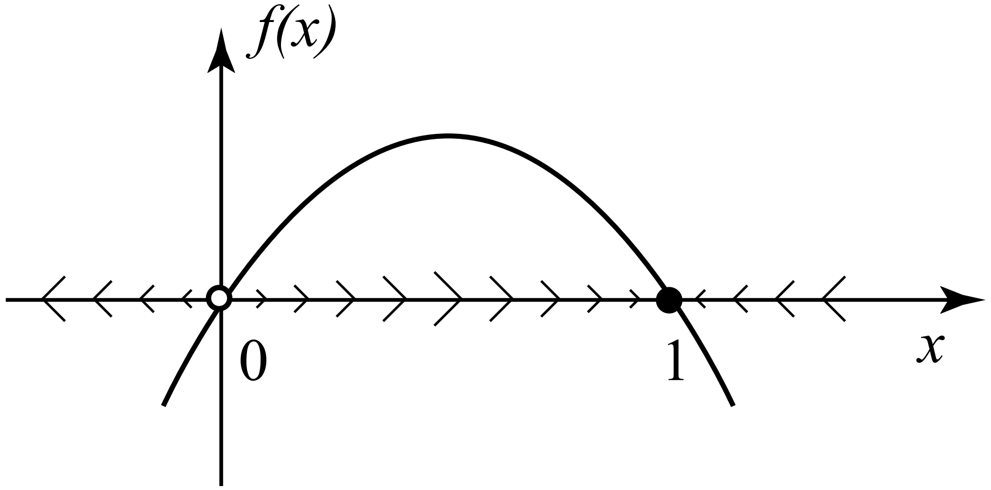

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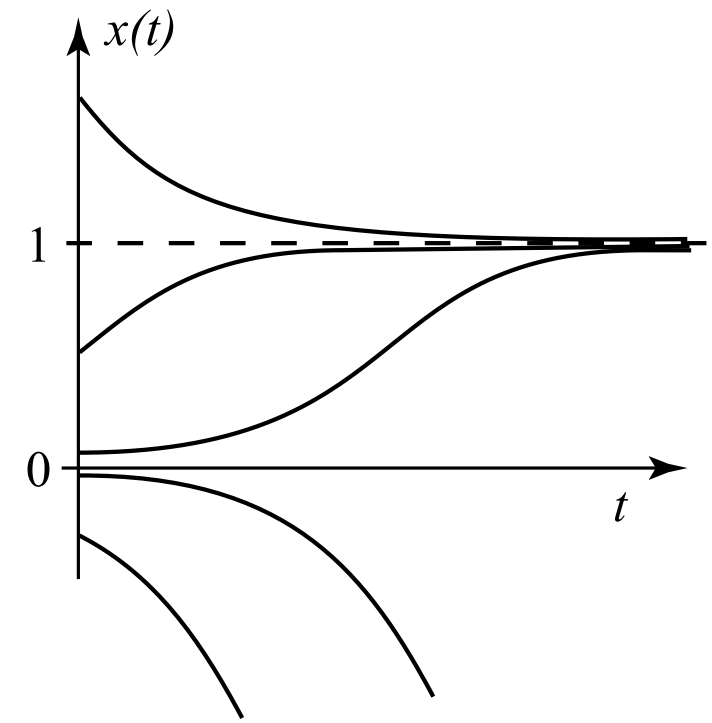
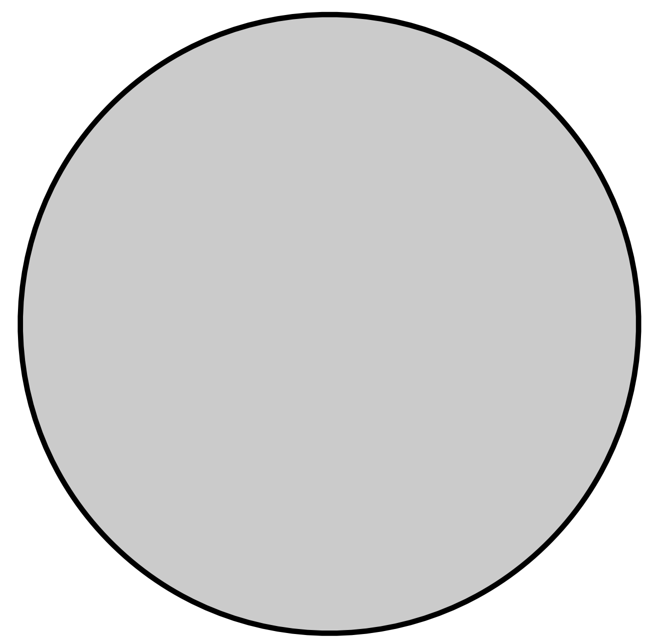


- The former corresponds to steady states (so not inflection points).

- The latter gives inflection points. This means maxima of $f(x)$ tell you the value of x at which inflections points of $x(t)$ occur.

Toward a drag equation

- What is the force require to move an object through a distance its own size?



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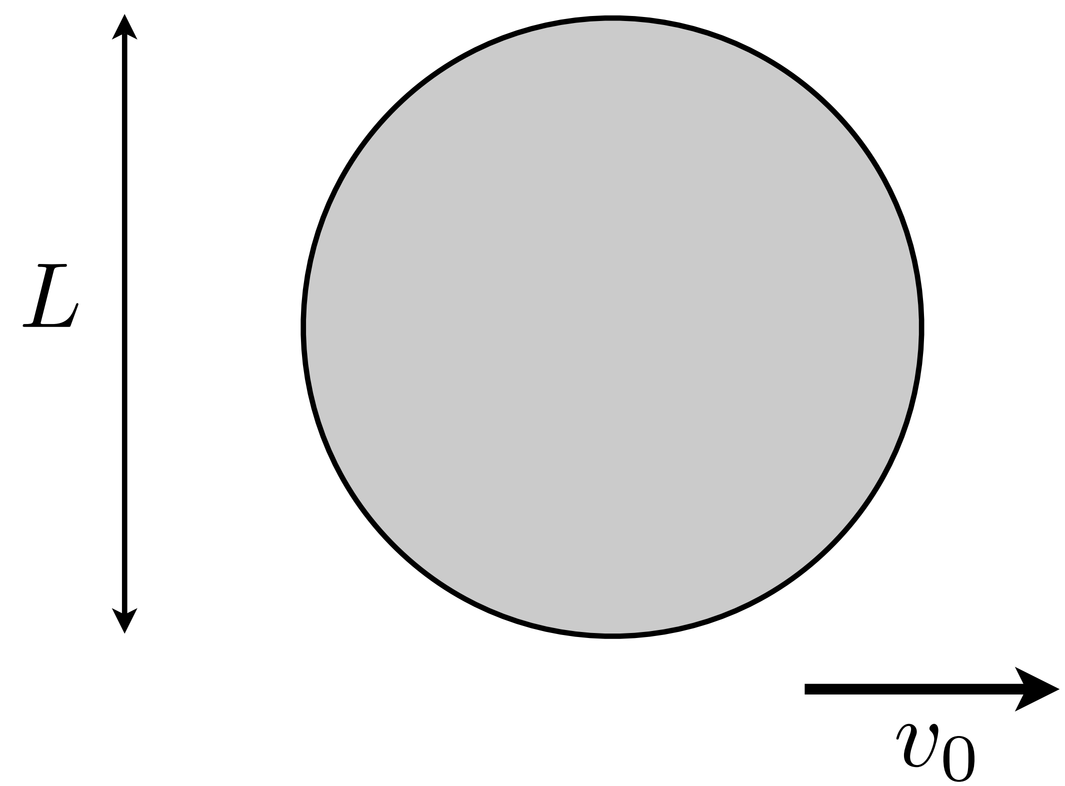

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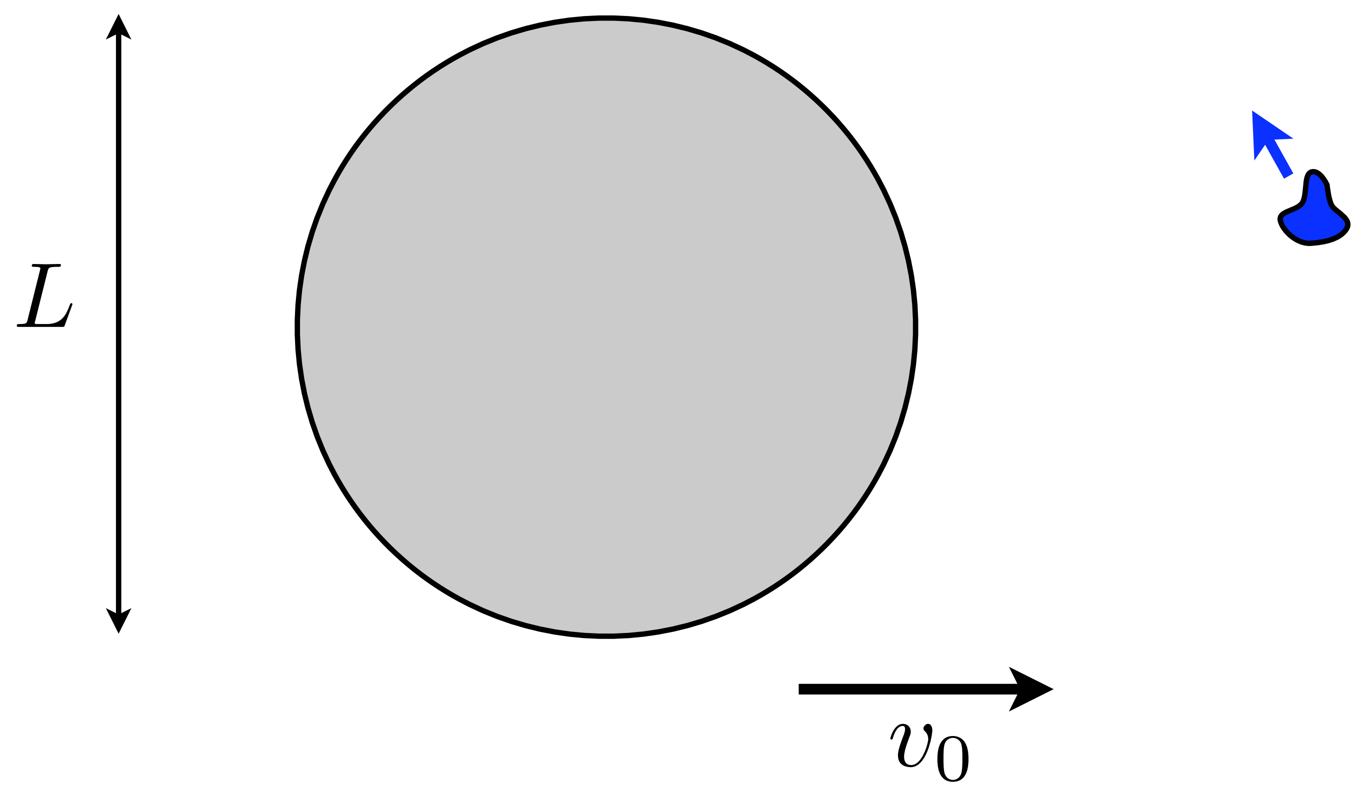

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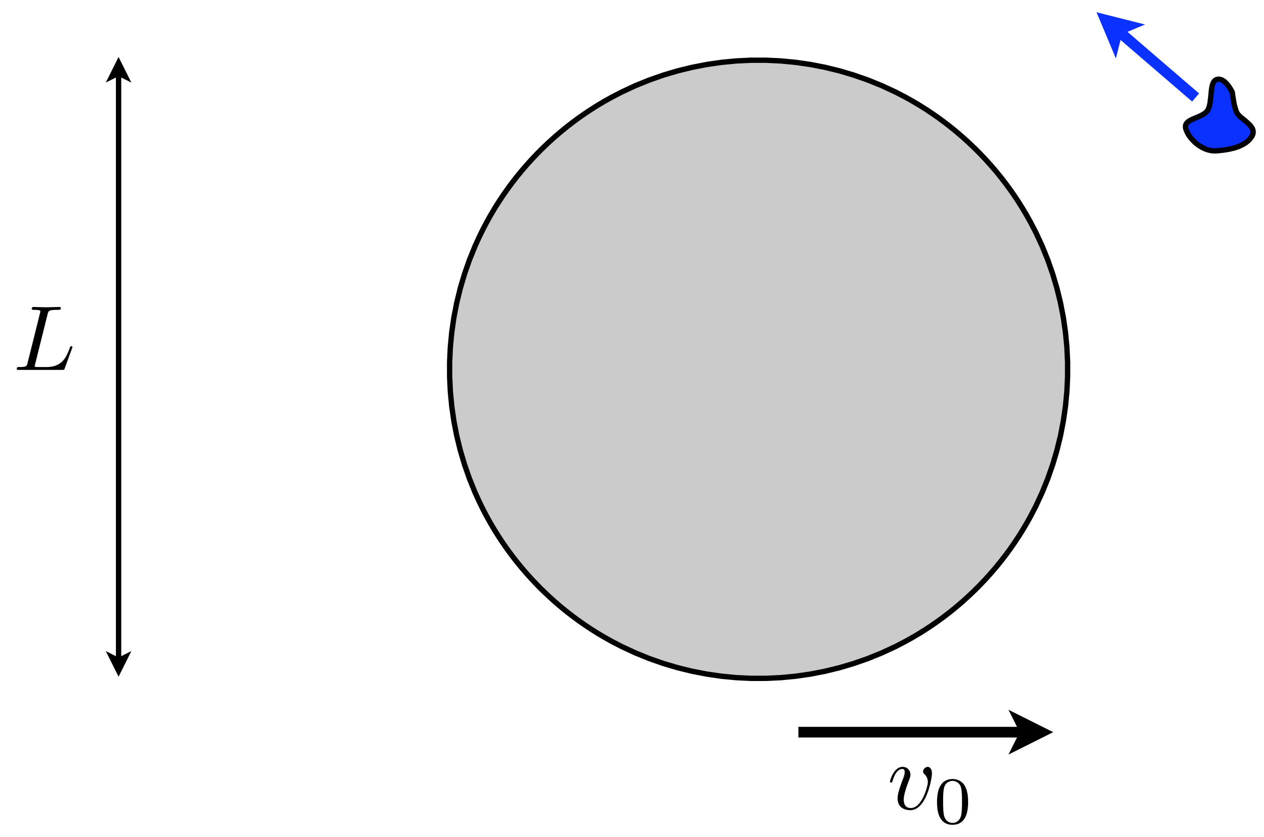

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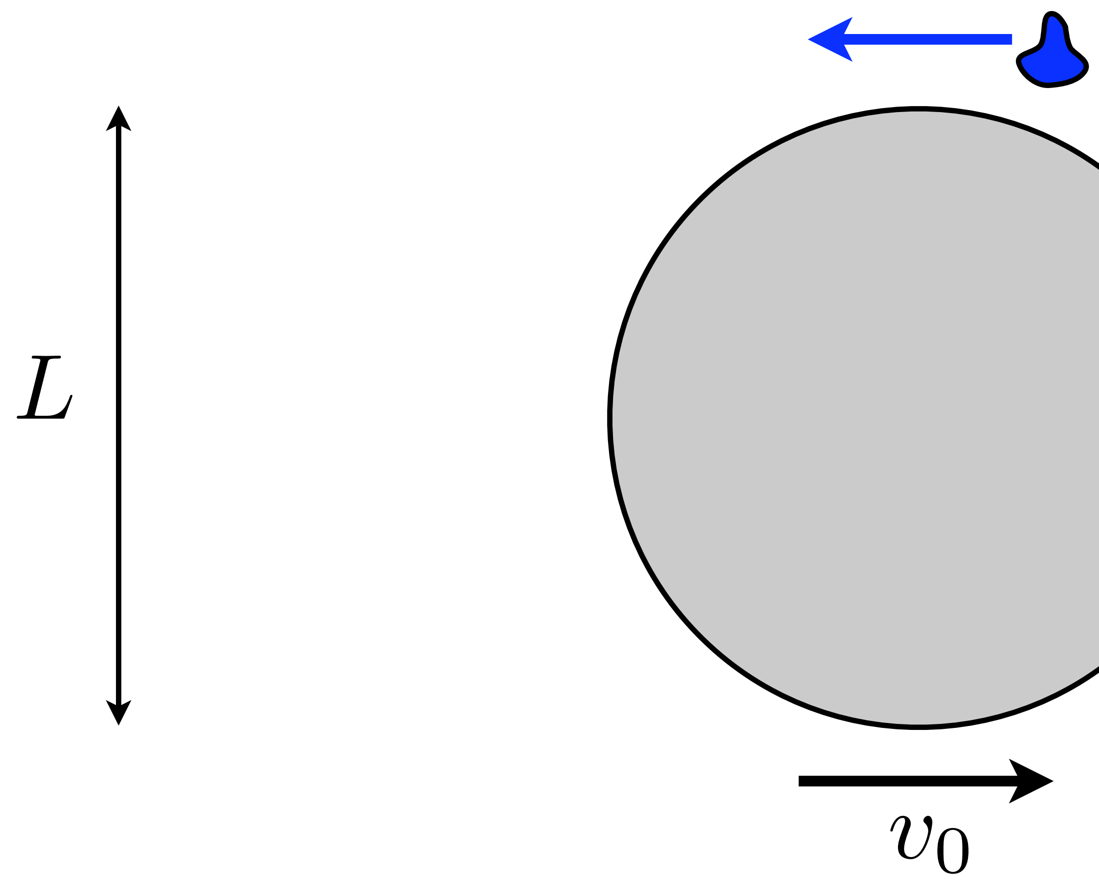

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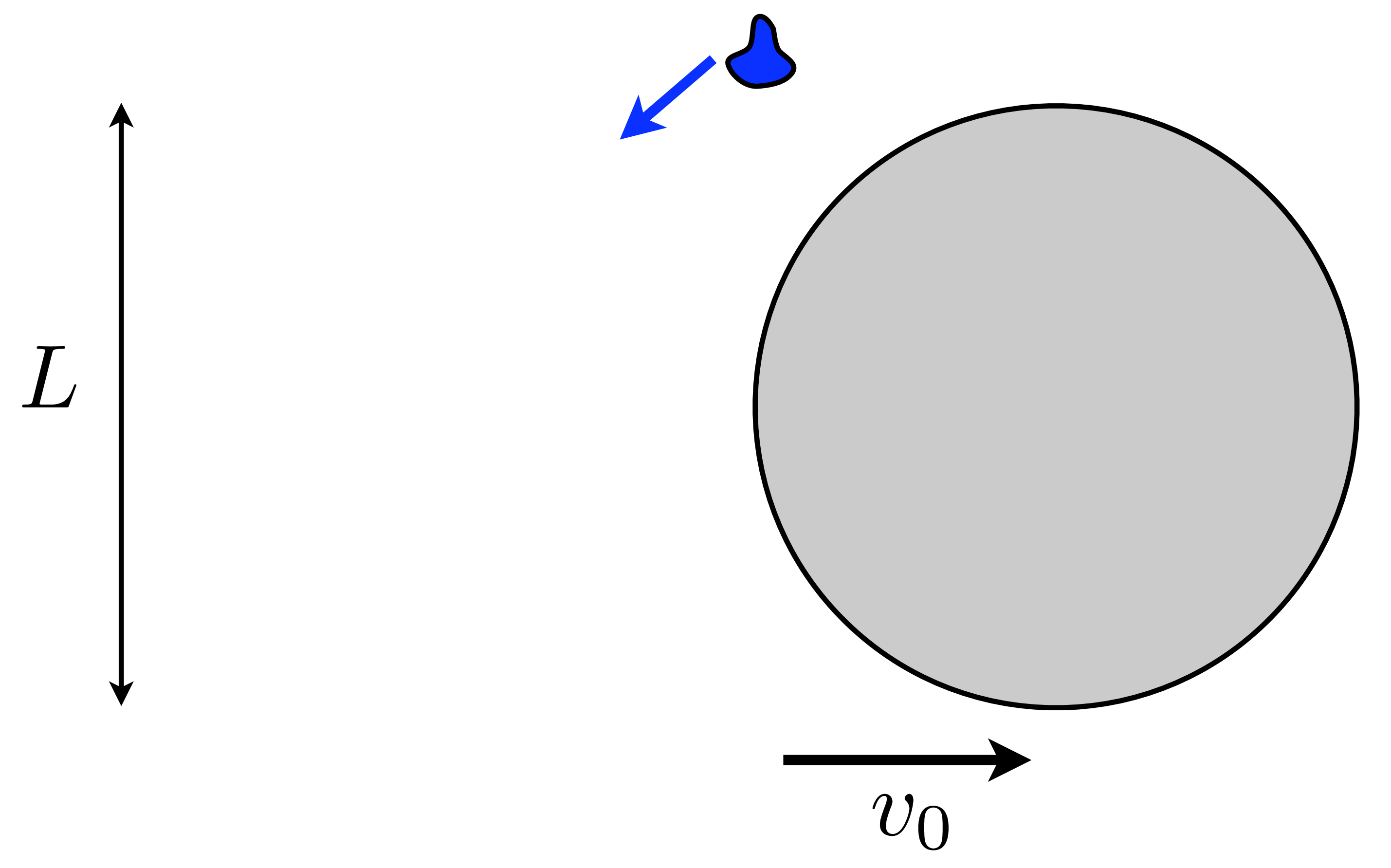

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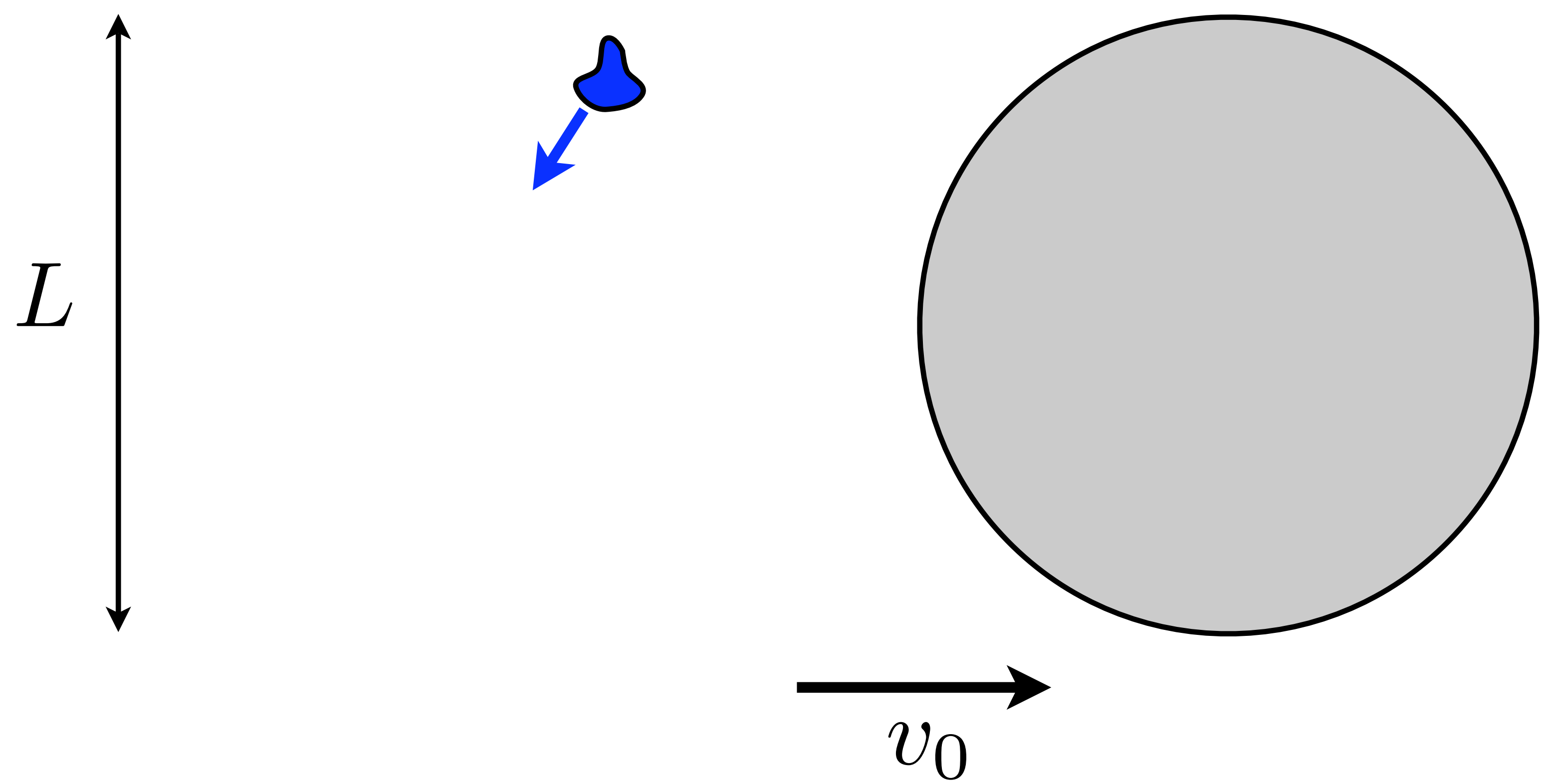

- Dimensional analysis to get drag force in inertial limit.

Toward a drag equation

- The drag force should depend on the object's size L (m) and velocity v (m s^{-1}), and the fluid's density ρ (g m^{-3}) and viscosity η (g m^{-1}s^{-1}).

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- C_D could account for this case provided it depends on viscosity.

Toward a drag equation - a dimensionless quantity

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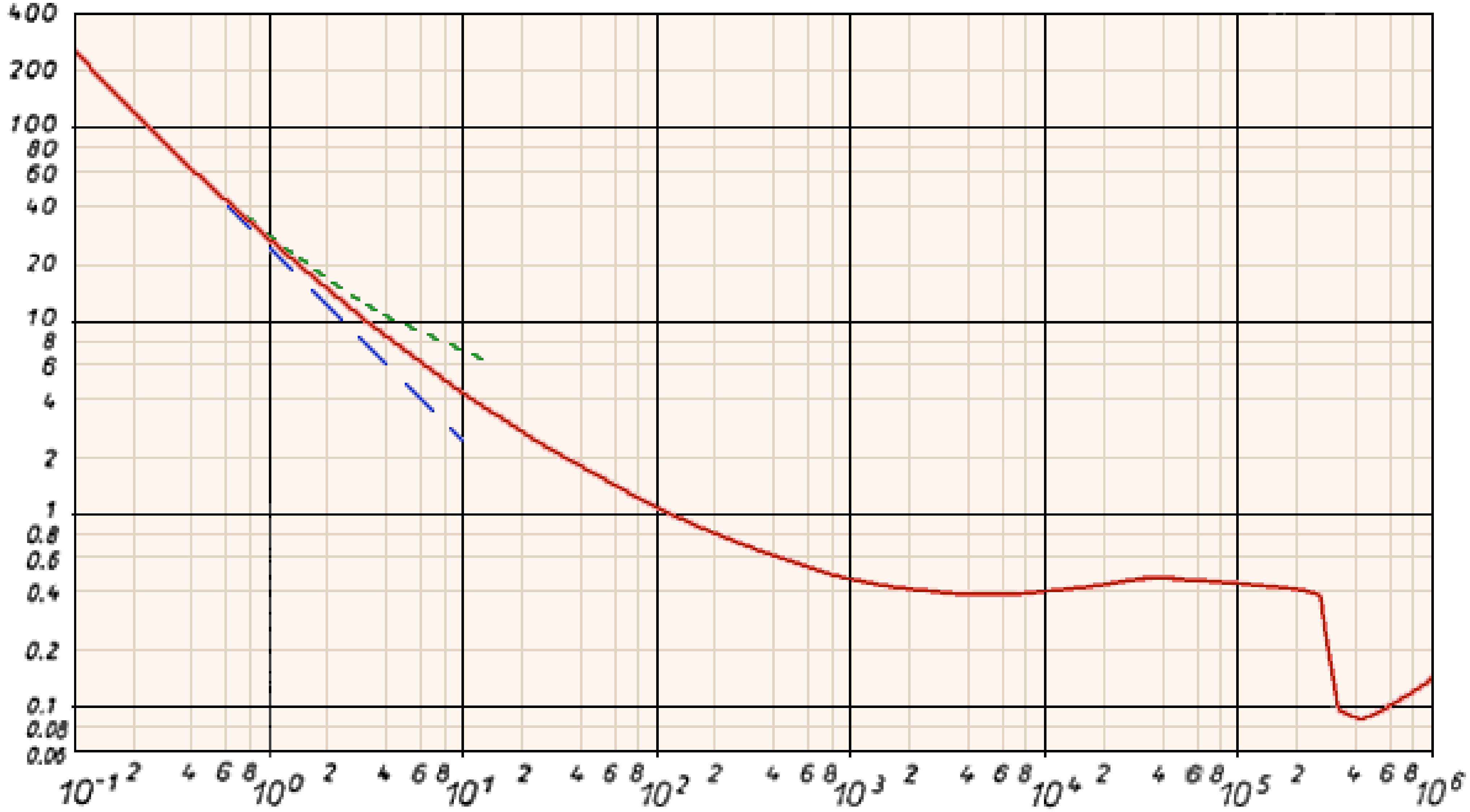
Drag on a sphere

$$C_D(Re) = -\left(\frac{C_1}{Re} + C_2\right)$$

C$_D$



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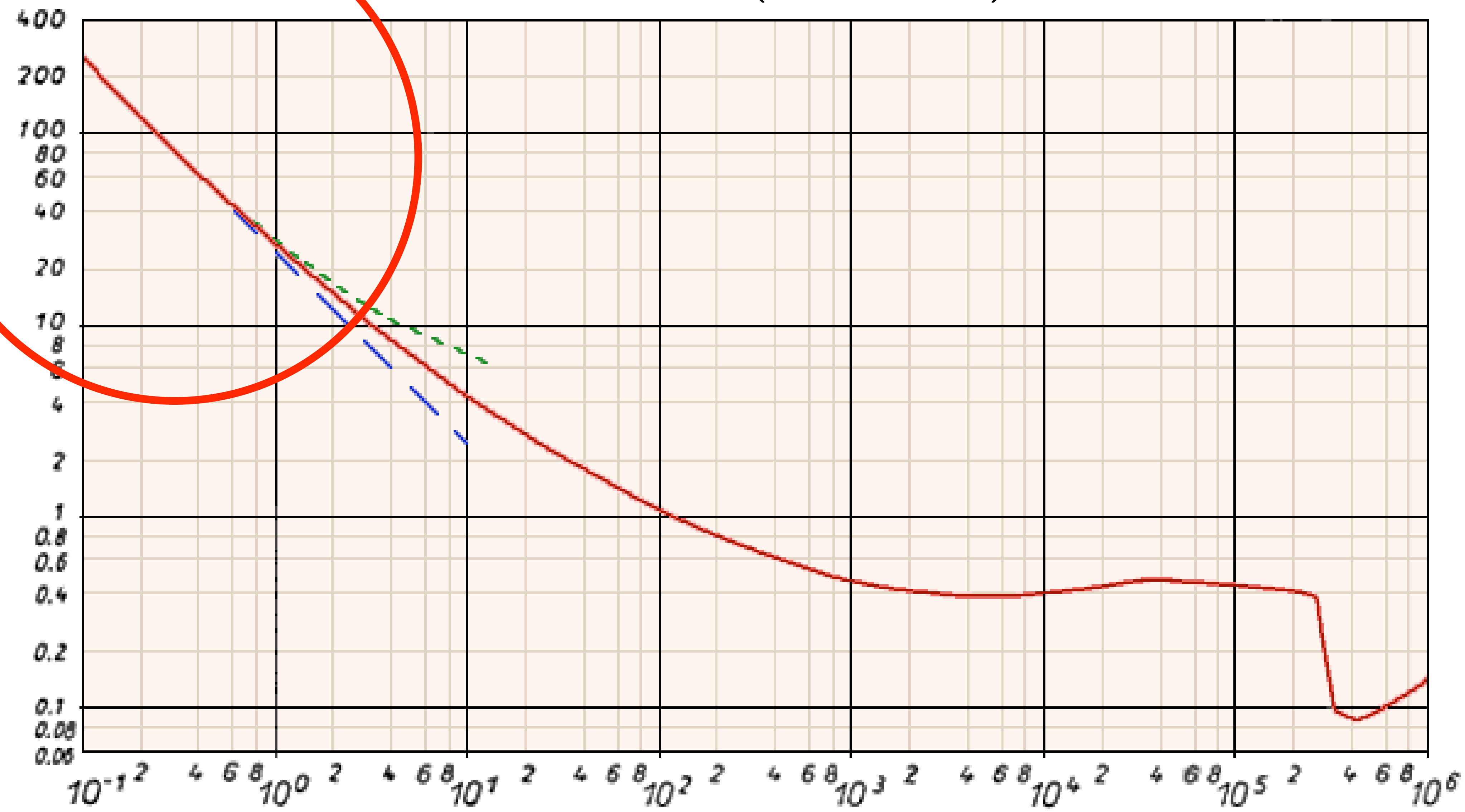

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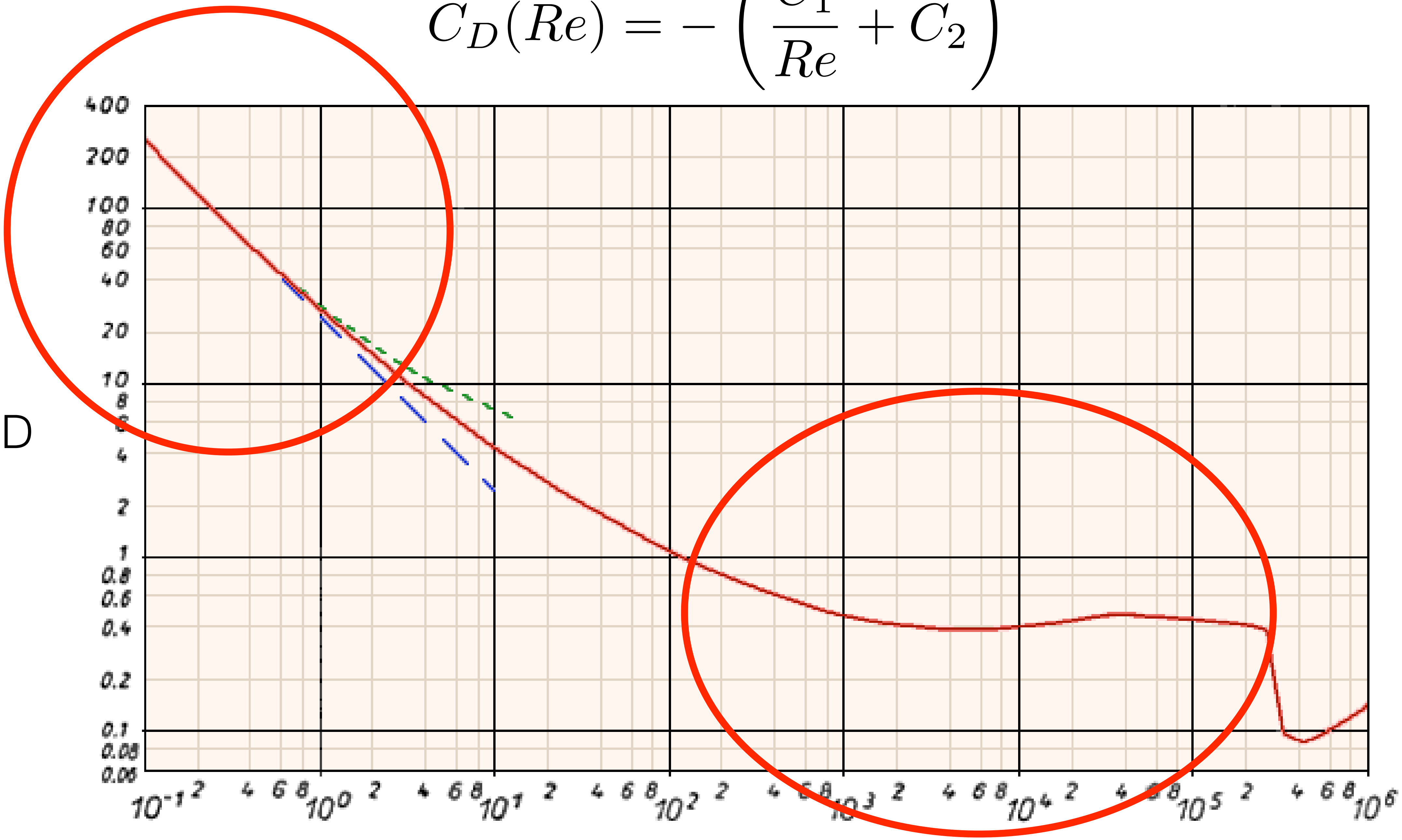

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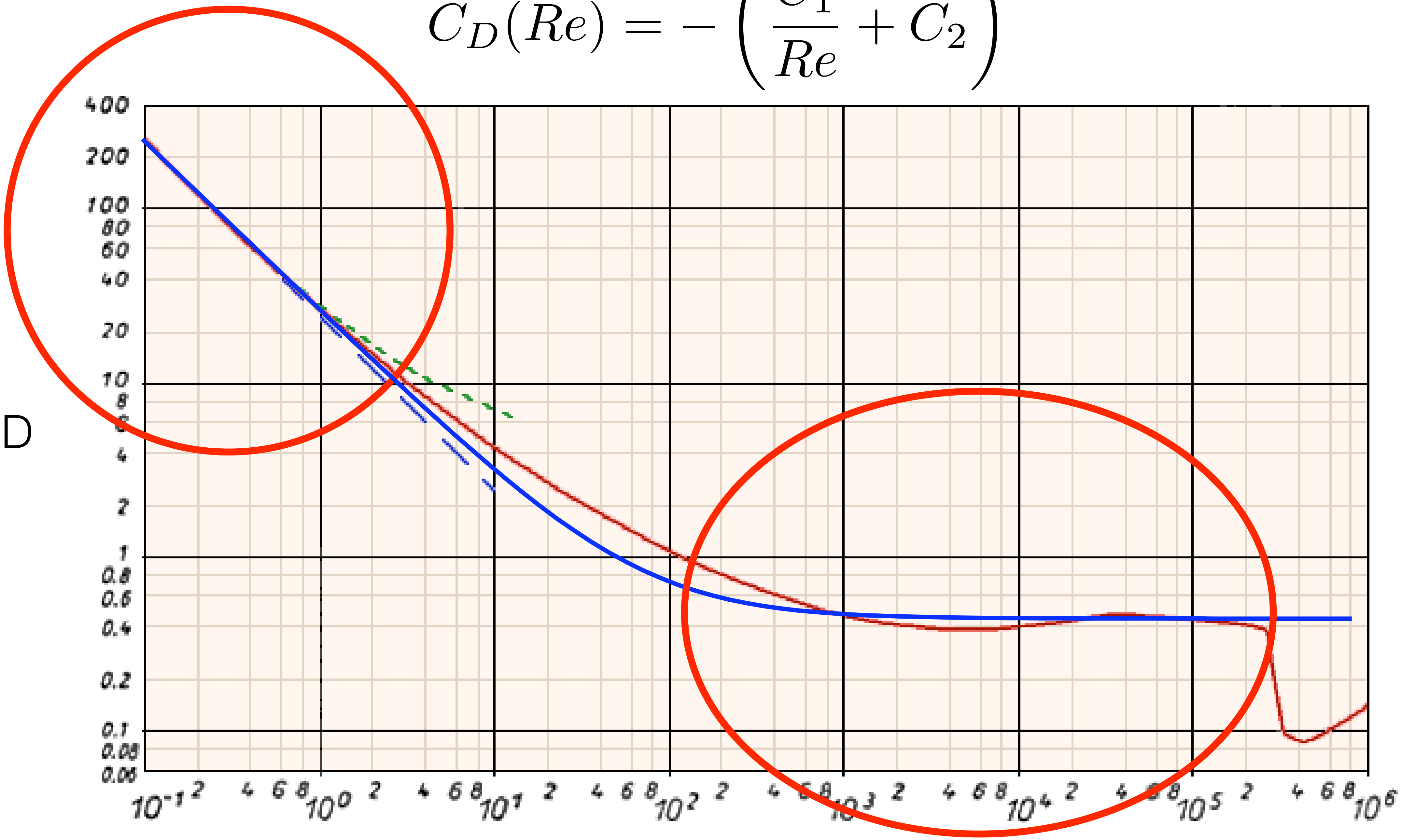

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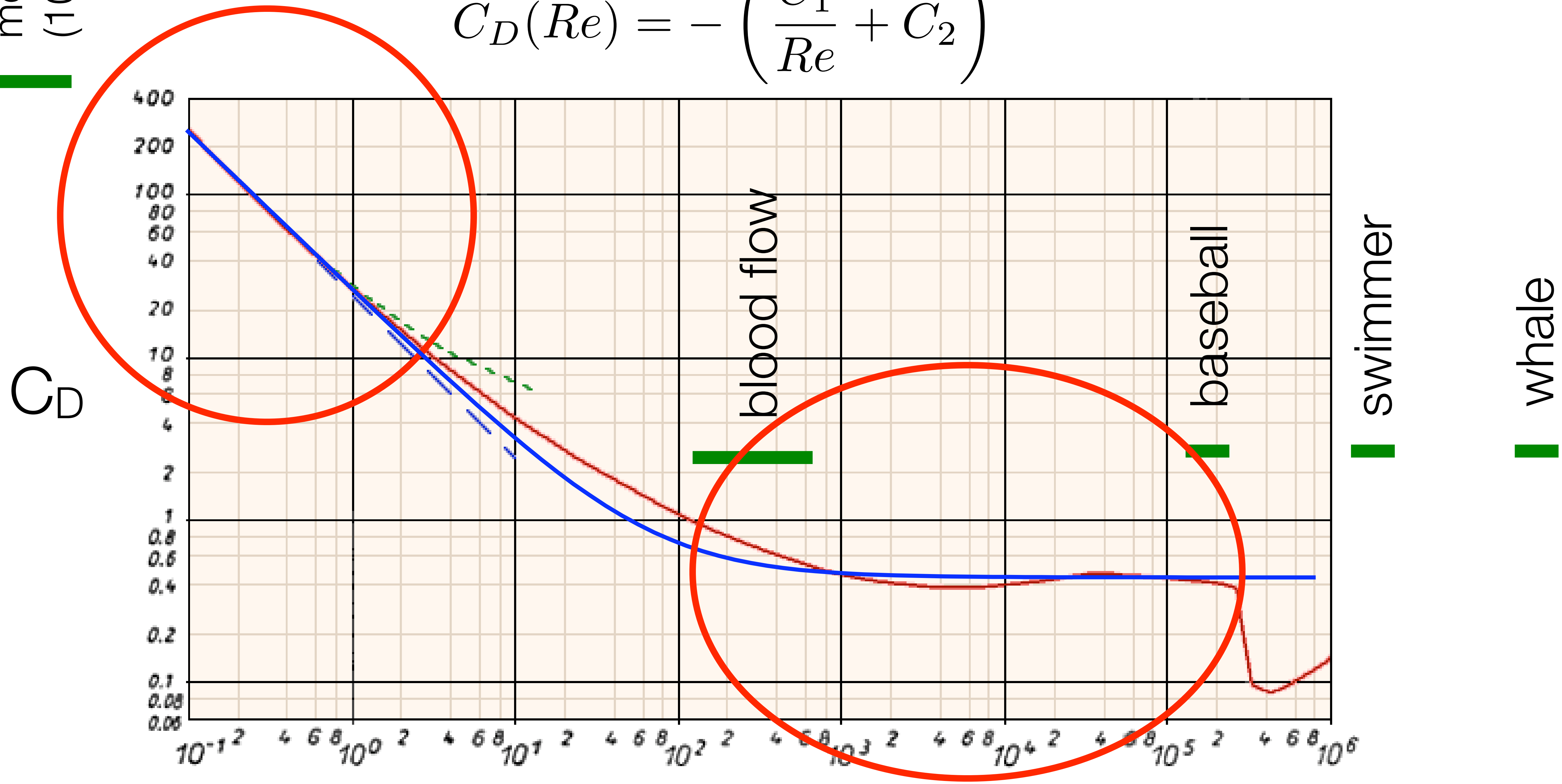

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Straight line; log-log plot (omitted in class)

Suppose v and w are the log-log plot coordinates and the curve is a line:

$$w = mv - b.$$

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Stokes predicted this; in particular, for a sphere, $C_0 = 6\pi$.

Aristotle’s bad rap

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Newton and Stokes would say:

$$ma = -\mu v + F_{other}.$$

When m is really small, $0 = -\mu v + F_{other}$

which is essentially what Aristotle said.

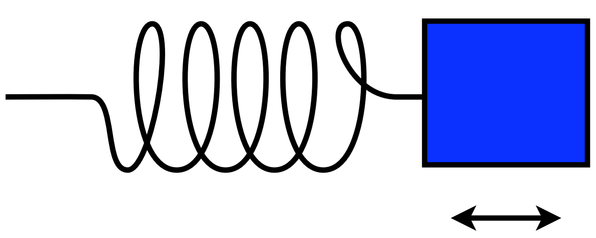
Freshman physics at low Reynolds Number

$$\mu \frac{dx}{dt} = F_{net}(x).$$

- When the net force is a function of position, we end up with a "first order differential equation".

- Examples:

$$\frac{dx}{dt} = -\frac{1}{\mu}\ k_{spring}\ x$$



$$\frac{dx}{dt} = -\frac{1}{\mu}\ \frac{k_{elec} q_1 q_2}{x^2}$$