MATHEMATICS: QUIZ – Curve Sketching

An outline of a complete solution to question 3.

For which values of c is the function $g(x) = \dfrac{x}{1+x^2} + cx$ increasing everywhere on the real line?

Solution:
In order to ensure that g is increasing everywhere, we must choose a value of c greater than the absolute maximum of $-f'$ (where f is defined by $f(x) = \frac{x}{1+x^2}$). This works because for such a c, we know that $c > -f'(x)$ for all values of x and so $g'(x) = f'(x) + c > 0$ for all values of x.
Procedure:

1. Find local extrema of f'.
2. Identify which extrema are minima using the second derivative test.
3. Find the value of $f'(x_{min})$ for each minimum.
4. Ensure there are no (negative) vertical asymptotes hiding in f'. If there are any such asymptotes, there is no value of c that will work.
5. Check for horizontal asymptotes of f' that may be below the lowest local minimum.
6. Check that $\lim_{x\to\pm\infty} f'(x) \neq -\infty$. This would cause the same problem as a negative vertical asymptote.
7. Choose a value of c so that $-c$ is lower than the lowest local minimum and lower than any horizontal asymptotes.