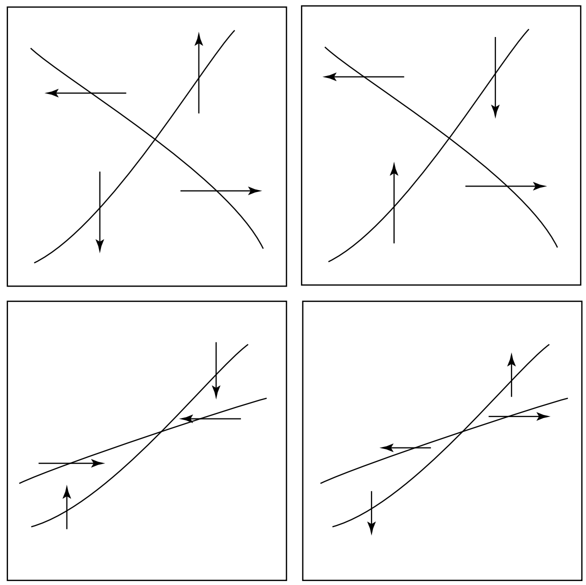
Problems on phase planes

1. In the figure below, draw a few solution curves and try to determine whether the steady state is stable or unstable. The solid curves are nullcines and the arrows indicate the direction in which solutions must cross the nullclines.



2. Consider the "mass-spring"-like equation

$$x'' - \alpha x' + \beta x = 0.$$

where α and β can take on any real values. Assume some solution has the form $x(t) = \exp(rt)$ and find both possible values of r that work (call them r_1 and r_2 so that $r_1 < r_2$). The general solution has the form $x(t) = A\exp(r_1 t) + B\exp(r_2 t)$. For each of the following cases, give conditions on α and β under which the case is true. For each one, state whether the steady state ($x(t) = 0$) is stable or unstable.

   (a) There is no oscillatory behaviour in any solution and all solutions decay to $x = 0$.

   (b) There is no oscillatory behaviour in any solution and all solutions grow exponentially except when $B = 0$ in which case the solution decays to $x = 0$.

   (c) There is no oscillatory behaviour in any solution and all solutions grow exponentially.

   (d) All solutions oscillate and grow exponentially.

   (e) All solutions oscillate and decay to $x = 0$.

3. Sketch the phase plane for each of the following systems. Sketch nullclines, direction vectors and steady states. Determine stability of each steady state by linearizing the system about it and using the conditions on $\alpha = a + d$ and $\beta = ad - bc$ that you derived above. The values of a, b, c and d appropriate to a steady state (x_0, y_0) are found by linearly approximating the functions on the RHS of each equation as follows:

$$x' = f(x, y) \approx a(x - x_0) + b(y - y_0),$$

$$y' = g(x, y) \approx c(x - x_0) + d(y - y_0).$$

Note that this system can be turned into a mass-spring-like system by substituting $v = x - x_0$ and $w = y - y_0$ and manipulating the resulting equations for v and w into a single second order equation.

(a)

$$\frac{dx}{dt} = 1 - x^2 - y^2$$
$$\frac{dy}{dt} = x - y.$$

(b)

$$\frac{dx}{dt} = y - x^2$$
$$\frac{dy}{dt} = x - y^2.$$

(c)

$$\frac{dx}{dt} = -x/2 + \sin(x) - y$$
$$\frac{dy}{dt} = x.$$

(d)

$$\frac{dx}{dt} = v$$
$$\frac{dv}{dt} = -px - qv$$

where p and q are both positive constants. What physical system does this model? Try converting it to a second order equation.

(e)

$$\frac{d\theta}{dt} = \omega$$
$$\frac{d\omega}{dt} = -p\sin(\theta) - q\omega$$

where p and q are both positive constants. What physical system does this model? Try converting it to a second order equation.

(f)

$$\frac{dx}{dt} = -x + py$$
$$\frac{dy}{dt} = 1 + q\frac{x^2}{1 + x^2} - ry.$$

where $p = (2+q)/2r$ and q and r are positive constants. Note that $x = 1, y = (2+q)/2r$ is one steady state.

4. For each of the systems above, pick a point in the phase plane to use as an initial condition and sketch the corresponding solution as a curve in the phase planes and then as functions of time (i.e. plot x versus t and y versus t).