Problems on phase lines

Although phase line analysis is most useful for equations that cannot be solved in closed form, and the following questions are all focussed on phase line arguments, many of the equations below *are* nonetheless solvable. For example, separation of variables works on the quadratic ones.

1. Find all steady states of the following equations and determine stability for each one, first qualitatively by sketching the function and then quantitatively by checking the slope of the RHS at the steady state. Sketch several solutions in the x-t plane. Be sure to include steady states and mark the locations of all inflection points on non-steady state solutions. For each equation, how many solutions do you need to sketch to show essentially all possible solutions?
   (a) $dx/dt = x^2 - 2x - 3$.
   (b) $dx/dt = x^2 + 3$.
   (c) $dx/dt = x^2 - 2x + 1$.
   (d) $dx/dt = -x^3 + x - 2/(3\sqrt{3})$.
   (e) $dx/dt = -x^4 + 2x^2 + 1$.

2. Draw the phase line for the following equations for several values of the parameter a. Choose one value of a for every possible qualitatively different phase line. To define, "qualitatively different", if two values of a both give phase lines with three steady states, with the lowest and highest stable and the middle one unstable, then they have, qualitatively, the same phase line.
   (a) $dx/dt = a - x^2$.
   (b) $dx/dt = -ax + x^3$.
   (c) $dx/dt = a - x + x^3$.
   (d) $dx/dt = a + x - x^3$.

3. Consider the equation $dx/dt = -bx + x^3$. Note that for $b > 0$, $x = 0$ is a stable steady state. For values of x very close to $x = 0$, the x^3 term is insignificant compared to the $-bx$ term. With this in mind, what function does a solution $x(t)$ that approaches zero look like as it get close to zero?

4. Consider the equation $dx/dt = bx - x^3$. Note that for $b > 0$, $x = \sqrt{b}$ is a stable steady state. For values of x very close to $x = \sqrt{b}$, a phenomenon similar to the one explored in the previous problem occurs. With this in mind, what function does a solution $x(t)$ that approaches $\sqrt{b}$ look like as it get close to $\sqrt{b}$? Replacing $f(x) = bx - x^3$ by a linear approximation about $\sqrt{b}$ will be useful.

5. Show that $x(t) = x_0 e^{kt}$ solves the equation $dx/dt = kx$ subject to the initial condition $x(0) = x_0$.

6. Show that any other solution to the equation $dx/dt = kx$, for example $y(t) = y_0 e^{kt}$, when shifted horizontally by a time t_0, is identical to $x(t)$ above, provided x_0 and y_0 have the same sign. Calculate t_0. Where does the condition on the sign arise?

7. Show that $x(t) = K/(1 + Ae^{-rt})$ solves the equation $dx/dt = rx(1 - x/K)$ (with $r, K > 0$) subject to the initial condition $x(0) = x_0$ where $A = (K - x_0)/x_0$ and $x_0 \neq 0$. What is the solution when the initial condition is $x_0 = 0$?

8. Show that any other solution to the equation $dx/dt = rx(1 - x/K)$, for example $y(t) = K/(1 + Be^{-rt})$, (where $B = (K - y_0)/y_0$) when shifted horizontally by a time t_0, is identical to $x(t)$ above, provided x_0 and y_0 are either both negative, both between 0 and K or both above K. Calculate t_0. Where does the condition on x_0 and y_0 arise?

9. Draw the phase line and sketch a few solutions (N versus t) for the second order chemical kinetics equation $dN/dt = -kN^2$.

10. Use the method of separating variables (described on slides 11-26 of lecture 22) to solve the second order chemical kinetics equation $dN/dt = -kN^2$. Does your answer agree with the phase line approach in the previous problem?