Modeling problems

Instructions: Work in groups of 3-4. Choose 1-3 of the following questions to work on for the next couple classes. If you feel you have a pretty good answer, select a spokesperson from your group. If there is time, we will have presentations later in the week.

1. Suppose that a population of fish, measured as N in thousands of individuals, has a natural growth rate that depends on its density as described by the function

$$G(N) = N(N-1)(3-N).$$

If the population size is N = 1 thousand and a few fish are removed, do you predict that the population size will recover to 1? If not, what does it do? What if the population size is N = 3 thousand - will it recover from a similar removal of a few fish?

For a population originally at N = 3, if you remove fish at a rate EN so that the total rate of change of the population is now determined by the natural growth rate G(N) and the fishing rate EN, find the new stable steady population. Note that this population size is a function of E; call it $N_{ss}(E)$. Larger E means more fishing and hence a smaller stable steady population - check that your expression for $N_{ss}(E)$ decreases with larger E as one might intuit. What is the upper limit on E for which there is a stable steady population? What happens above this value of E?

What is the yield (fish taken per unit time) when fishing occurs with rate constant E? Find the fishing rate constant, E_{max} at which the yield is largest. What is the maximum sustainable yield (i.e. maximum yield that does not cause the population to crash)? Check that your answer for E_{max} is within the allowable range.

Try to provide a graphical interpretation of your answers.

- 2. Suppose you have just taken a spherical allergy pill with radius r_0 . The volume of the pill gradually decreases as your stomach digests it but it remains spherical throughout. Suppose the rate of change of volume (V) of the remaining portion of the pill is proportional to its surface area (S) with rate constant k. Derive an equation for the radius of the pill as a function of time. Solve the equation and calculate the length of time required to completely digest the pill or if more appropriate the time constant associated with the pill's dissolving.
- 3. A microtubule (MT) is a linear polymer that grows by addition of tubulin subunits at its tips. Tubulin subunits are placed in a rectangular microfabricated chamber (a micron-scale hole etched out of glass) with dimensions $2\mu m \times 8\mu m \times 8\mu m$ at an initial concentration of $5\mu M$. No microtubules are initially present. At a particular moment in time, six MTs spontaneously form, each with a negligible (essentially zero) length. From that time on, tubulin continually adds to and falls off the two tips of each polymer – the rate of addition at each tip is proportional to the tubulin concentration with rate constant $k_{on} = 5$ subunits/(second· μM) and the rate of removal is a constant rate $k_{off} = 20$ subunits/second. Some assumptions that will be useful: (1) Addition of a single tubulin subunit increases the length of the polymer by approximately $\delta = 0.5$ nm. (2) Tubulin subunits are conserved meaning that their total number never changes; they simply change from soluble form (measured in concentration) to polymer form (measured as length) and back. (3) Avogradro's number= 6.02×10^{23} . (4) Although addition of subunits increases the length in discrete increments, it is reasonable to treat length as a continuous variable.

- (a) Write down a differential equation for the total length of polymer as a function of time, l(t). What is the appropriate initial condition? This question requires you to think carefully about conversion of units!
- (b) What is the total polymer length after a long period of time? Do the polymer fit in the chamber without having to bend?
- (c) What must I mean in the previous question when I say "a long period of time"? That is, a long time compared to what?
- (d) Simplify and solve the equation.
- 4. (a) Propose a function that would be good model for the amount of rainfall in Vancouver per day per square meter over a typical year. If you stuck a big bucket on the roof, how much water would be in it after one year (assume no evaporation)?
 - (b) Propose a function that would be good model for the amount of sun in Vancouver per day per square meter. If you stuck a solar panel on the roof, how much energy would you collect during one year?
 - (c) Plants need both sun and water in order to grow. Use your answers to the first two parts to construct a simple model for plant growth rate. According to your model, when would the plants undergo maximum growth? Propose a differential equation model for a plant's height. How much growth does your model predict in one year?
 - (d) Repeat this for a another location with different rainfall and sun data. Discuss the impact of the differences.
 - (e) Pose a question regarding the role of bears and salmon in forest ecology along the BC coast that can be addressed using a modified version of the model above. Write down a model that could be used to answer the question.
- 5. Consider the heat loss of an arctic hare compared to a desert hare. Assume that body temperature increases as a result of metabolic activity in every cell and that heat is lost across a hare's body surface.
 - (a) Propose a differential equation model for body temperature as a function of time for both arctic and desert hares. Assume for now that the hares are being kept in a lab at room temperature. Does your model support the claim that a desert hare's ears make a significant impact on its ability to shed excess heat?
 - (b) Modify your model to account for hares in the wild (daily temperature variation etc.).
 - (c) Use a spreadsheet to solve the wild-hare equations numerically.
- 6. Assume that the force a kinesin motor can generate when moving at velocity v is given by

$$F_{kin}(v) = F_{kin}^0 \left(1 - \frac{v}{v_{kin}}\right).$$

Assume that the force a dynein motor can generate when moving at velocity v is given by

$$F_{dyn}(v) = -F_{dyn}^0 \left(1 + \frac{v}{v_{dyn}}\right).$$

Assume the reference frame has been chosen so that positive velocities indicate leftward movement and positive forces point leftward.

- (a) How much force does a kinesin motor generate when moving at v_{kin} ?
- (b) How much force does a kinesin motor generate when not moving at all?
- (c) In what direction is a kinesin motor moving when the motor is not producing any force (i.e. unencumbered by a load).
- (d) In what direction is a dynein motor moving when the motor is not producing any force.
- (e) Suppose a kinesin motor is carrying a protein along a microtubule. Write down the force balance equation for the motor's velocity. Solve the equation for v(t). Estimate the mass and drag coefficient for a typical protein (assume it consists of 200 amino acids and is roughly spherical). Calculate the characteristic time it takes for the protein to reach its terminal (i.e. steady state) velocity if it is initially at rest. How far does the protein move in 10 seconds? What is the terminal velocity? How does it compare to v_{kin} ? Interpret your answers to these questions.
- (f) The diffusion coefficient of a particle is related to its drag coefficient by the so-called Einstein relation $D = k_B T/\mu$ where k_B is Boltzmann's constant and T is absolute temperature (assume roughly body temperature). Recall (or rederive) the dimensional estimate for the time required for a particle to travel a distance Δx . A neuron must ensure that vesicles full of neurotransmitters reach the far end of an axon as quickly as possible. Is it faster to wait for diffusion to deliver the vesicles or to send them along microtubules driven by kinesin motors? Your answer should depend on the length of the axon L.
- (g) Suppose a chromosome is attached to a microtubule by N kinesin motors and M dynein motors. Calculate the steady state velocity of the chromosome. Does the drag force play an important role? In the limit that N >> M, what velocity does the chromosome approach?
- 7. You are interested in understanding some simple facts relating to body temperature and the risk of hypothermia. Write down a differential equation model (or models) and use it (them) to calculate the following functions of time.
 - (a) The body temperature of a person swimming (constant exertion) in the ocean.
 - (b) The body temperature of the same person floating on her back in the same ocean.
 - (c) The body temperature of the same person curled into a ball (wearing a life-vest).
 - (d) The body temperature of two people huddled together in the same ocean.

Discuss the meaning of steady states and time constants in terms of hypothermia. What are the physical differences between each of the cases and how do those differences translate into parameter values? Discuss the connection between physical differences, parameters values and the steady states and time constants.

Note that all of the individuals mentioned above are alive. When out of the water and in a relatively warm environment, your model should predict a typical human body temperature.