Linearizing about a steady state - an example

Consider the system

$$\frac{dx}{dt} = \cos(x-y),$$
$$\frac{dy}{dt} = \sin(x+y).$$

This system has steady states when $x-y = \pi/2 + n\pi$ for any integer value of n and $x+y = m\pi$ for any integer value of m. Because of periodicity, all steady states will look like one of the four steady state associated with $n = 0, 1$ and $m = 0, 1$. To demonstrate how to linearize about a steady state, I'll consider the case for $n = 1, m = 1$. For these values, there is a steady at the intersection of the lines $x - y = 3\pi/2$ and $x + y = \pi$, that is, at $x = 5\pi/4,\ y = -\pi/4$. To determine the stability of this steady state, we must linearize $\cos(x-y)$ and $\sin(x+y)$ about the point $(5\pi/4, -\pi/4)$. Let's give the functions names for notational convenience; $f(x,y) = \cos(x-y)$, $g(x,y) = \sin(x+y)$.

$$f(x,y) \approx f(5\pi/4, -\pi/4) + \frac{\partial f}{\partial x}(5\pi/4, -\pi/4)(x - 5\pi/4) + \frac{\partial f}{\partial y}(5\pi/4, -\pi/4)(y + \pi/4).$$

Because $(5\pi/4, -\pi/4)$ is a steady state, $f(5\pi/4, -\pi/4) = 0$. Next, recall that $\partial f/\partial x$ is the derivative of f with respect to x while holding y constant. So $\partial f/\partial x = -\sin(x-y)$. And $\partial f/\partial y = \sin(x-y)$. Next, we evaluate these at the point $(5\pi/4, -\pi/4)$:

$$\frac{\partial f}{\partial x}(5\pi/4, -\pi/4) = -\sin(3\pi/2) = 1, \qquad \frac{\partial f}{\partial y}(5\pi/4, -\pi/4) = \sin(3\pi/2) = -1.$$

Similarly, we need to calculate the partial derivatives of $g(x,y)$; $\partial g/\partial x = \cos(x+y)$, $\partial g/\partial y = \cos(x+y)$:

$$\frac{\partial g}{\partial x}(5\pi/4, -\pi/4) = \cos(\pi) = -1, \qquad \frac{\partial g}{\partial y}(5\pi/4, -\pi/4) = \cos(\pi) = -1.$$

Now we have everything we need to write down the linearization of the original system about the steady state $(5\pi/4, -\pi/4)$:

$$\frac{dx}{dt} = (x - 5\pi/4) + (-1)(y + \pi/4),$$
$$\frac{dy}{dt} = (-1)(x - 5\pi/4) + (-1)(y + \pi/4).$$

To clean this up a bit, we can shift the axes so that the steady state is at the origin (of a new set of axes) by using the substitutions $u = x - 5\pi/4$ and $v = y + \pi/4$. The resulting system is

$$\frac{du}{dt} = u + (-1)v,$$
$$\frac{dv}{dt} = (-1)u + (-1)v.$$

Stability can be checked by evaluating $\alpha = a + d$ and $\beta = ad - bc$ where $a = 1, b = -1, c = -1$ and $d = -1$. These give $\alpha = 0$ and $\beta = -2$. Thus, the associated mass-spring-like equation is $u'' - 2u = 0$. The solution to this equation is $u(t) = A\exp(-\sqrt{2}t) + B\exp(\sqrt{2}t)$ so there are many initial conditions for which the solution grows exponentially and a few (carefully chosen) solutions for which $u(t)$ approaches zero (i.e. those for which $B = 0$). This kind of steady state is unstable and is called a saddle. Assuming that the approximate system has solutions that are like the solutions to the original nonlinear system near the steady state $(5\pi/4, -\pi/4)$, we can conclude that this steady state has a similar structure (i.e. it's a saddle).