Problems on antiderivative and solving DEs

1. For each of the following functions, calculate its derivative.

   (a) $f(x) = x^3 + \cos(x)$.
   (b) $g(x) = \cos(x^3)$.
   (c) $h(x) = \sin(\ln(x))$.
   (d) $j(x) = \cos(y(x))$.
   (e) $k(x) = e^{\tan(x)}$.
   (f) $l(x) = e^{z(x)}$.
   (g) $m(x) = \ln(\cos(x))$.
   (h) $n(x) = \ln(x^2 + 4)$.

2. For each of the following functions, determine the entire family of functions, called antiderivatives, whose derivatives are exactly the given function. The term *family* is used whenever a function is expressed in terms of an arbitrary constant. For example, for the function $f(x) = x^2$, the family of antiderivatives is $F(x) = x^3/3 + C$ where C is an arbitrary constant. $F(x)$ is not just a single function since we can choose many different values of C and still get an antiderivative of $f(x)$. You should check your answer by taking its derivative to make sure you got it right.

   (a) $f(x) = x^2 + \sin(x)$.
   (b) $g(x) = 2x\cos(x^2)$.
   (c) $h(x) = \cos(\ln(x))\dfrac{1}{x}$.
   (d) $j(x) = \cos(y(x))\dfrac{dy}{dx}$.
   (e) $k(x) = e^{\cos(x)}\sin(x)$.
   (f) $l(x) = e^{z(x)}\dfrac{dz}{dx}$.
   (g) $m(x) = \dfrac{\cos(x)}{\sin(x)}$.
   (h) $n(x) = \dfrac{1}{x^3+2}\,3x^2$.
   (i) $p(x) = \dfrac{1}{x^3+2}\,x^2$.
   (j) $q(x) = \dfrac{1}{y(x)}\,\dfrac{dy}{dx}$.

3. Find the family of functions that solves each equation by rewriting the equation in the form $F(y(x))\cdot dy/dx = G(x)$, calculating antiderivatives of both sides and isolating $y(x)$. Remember that the antiderivatives of two functions that are equal to each other can differ by an arbitrary constant.

   (a) $\dfrac{dy}{dx} = x^2 y(x)$.
   (b) $y(x)\dfrac{dy}{dx} = x^2$.

(c) $\dfrac{dy}{dx} = -ky(x)$.

(d) $\dfrac{dy}{dx} = -ky(x)^2$.

(e) $\dfrac{dy}{dx} = -\tan(y(x))$.

(f) $\dfrac{dy}{dx} = \sec(y(x))$.

4. When an equation does not include the independent variable explicitly, it is possible to draw a phase line and sketch solutions for initial values $y(0) = y_0$ without actually solving the equation. Do so for those equations in the previous problem for which this is possible and compare your sketches to the solution you calculated by taking antiderivatives. Think about whether there are constraints on what initial conditions are allowable for each equation and whether solutions to initial value problems exist for all values of $x > 0$. For example, suppose the function $y(x) = \tan(x)$ is found to solve a particular differential equation subject to the initial condition $y(0) = 0$. Because this function blows up at $x = \pi/2$, it is only a solution to the equation on the interval $[0, \pi/2)$. Note that both the function $y(x)$ and its derivative dy/dx must be well defined and continuous for $y(x)$ to qualify as a solution.

5. Suppose $F(y)$ is the antiderivative of $f(y)$ so that $dF/dy = f(y)$. In addition, suppose $G(x)$ is the antiderivative of $g(x)$. Finally, assume that the function h is the inverse function of f, that is $h(f(y)) = y$.

   (a) What is the antiderivative of $f(y(x)) \cdot dy/dx$?

   (b) What is the antiderivative of $g(x)$?

   (c) Suppose that you are given information about the function $y(x)$ in the form of the equation
   $$\frac{dy}{dx} = \frac{g(x)}{f(y(x))}.$$
   Use this information together with the answer to (5a) and (5b) to determine a family of solutions $y(x)$ to the equation.