The University of British Columbia

Science One Mathematics

Midterm #1 – solutions

27 October 2009

Time: 50 minutes

Full Name: ______________________ Student # : ______________

Signature: ______________________

This Examination paper consists of 6 pages (including this one). Make sure you have all 6.

Instructions:

No memory aids allowed. No calculators allowed. No communication devices allowed.

Rules:

- Each candidate must be prepared to produce, upon request, a UBC card for identification.
- Candidates are not permitted to ask questions of the invigilators, except in cases of supposed errors or ambiguities in examination questions.
- No candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination.
- Candidates suspected of any of the following, or similar, dishonest practices shall be immediately dismissed from the examination and shall be liable to disciplinary action:
  - having at the place of writing any books, papers or memoranda, calculators, computers, sound or image players/recorders/transmitters (including telephones), or other memory aid devices, other than those authorized by the examiners;
  - speaking or communicating with other candidates; and
  - purposely exposing written papers to the view of other candidates or imaging devices. The plea of accident or forgetfulness shall not be received.
- Candidates must not destroy or mutilate any examination material; must hand in all examination papers; and must not take any examination material from the examination room without permission of the invigilator.
- Candidates must follow any additional examination rules or directions communicated by the instructor or invigilator.

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marking:

Q1	/	**Q3**	/	**Q2**	/	**Q4**	/	TOTAL	/

Names of Instructors: Eric, Fok, Mark

NOTE: these solutions are intended to provide a rough sketch of the correct answers. Details that were required for full points may be omitted in places.

Q1 [5 marks]

Let (a, a^2) be any point on the parabola $y = x^2$.

(a) Find the equation of the line, l_1, going through the point (a, a^2) and *perpendicular* to the tangent line to this parabola at (a, a^2).

Take derivative, plug in $x = a$ to get slope at (a, a^2). Slope of perpendicular is negative reciprocal so $m = -1/(2a)$. Equation of line is $y = a^2 + \frac{1}{2} - \frac{1}{2a}x$.

Common mistake: if you use x as the independent variable for the equation of the line AND for the point at which you evaluate the slope (which should really be a), you end up with something that is not the equation of a straight line.

(b) Consider the horizontal line l_2 given by $y = a^2$. Show that the distance between the y-intercepts of l_1 and l_2 is equal to $1/2$ for any non-zero value of a.

From the equation above, the y-intercept is $a^2 + 1/2$ so the distance between the two y-intercepts is clearly $1/2$.

Q2 [5 marks]

(a) Define what it means for a function f to be continuous at a point c.

See textbook for precise definition.

(b) Let $f(x) = \begin{cases} \sqrt{|x|} & \text{if } x < c, \\ x^2 & \text{if } x \geq c. \end{cases}$

For what value(s) of the constant c is f continuous everywhere.

Consider the two pieces as functions on the whole real line. Both of them are continuous so we need only worry about the point at which the switch occurs ($x = c$). To ensure continuity, c must be chosen at points where the graphs of the functions cross. So $c = -1, 0$ or 1 all work. This can be shown carefully by taking left and right limits and equating them to ensure continuity.

$$\sqrt{|c|} = \lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = c^2.$$

Square both sides and treat the $|\cdot|$ with two distinct cases. You should find the three values above solve this equation.

Q3 [10 marks]

Let $f(x) = \dfrac{1}{1+x^2}$.

(*a*) Identify intervals on which f is increasing, decreasing, concave up, concave down and identify any minima, maxima and inflection points of f.

Increasing for $x < 0$, decreasing for $x > 0$, concave up for $x < -1/\sqrt{3}$ and for $x > 1/\sqrt{3}$, concave down in between those values. Max at $x = 0$, inflection points at $\pm 1/\sqrt{3}$.

(*b*) Sketch f. Note: each axis must be labeled and must have at least one point indicated to provide a scale.

Imagine a sketch here - a bump at the origin with horizontal asymptotes at $\pm\infty$ going to zero. Axes should be marked at inflection points and maximum. The locations of marked inflection points should line up with concavity changes in your sketch.

(*c*) For which values of c is the function $g(x) = f(x) + cx$ increasing everywhere on the real line?

In order to ensure that g is increasing, c must be large enough to get rid of all negative slopes. Because $g(x) = f(x)+cx$, we know $g'(x) = f'(x)+c$ so g will be increasing provided c is larger than the negative of the most negative slope of f. The steepest negative slope of f occurs at the inflection point at $x = 1/\sqrt{3}$. So c must satisfy $c > -f'(1/\sqrt{3})$.

Q4 [10 marks]

Consider the function

$$f(x) = \frac{x^2 - 3x + 2}{x^2(4 - x)}.$$

Prove, using the Intermediate Value Theorem, that $f(x)$ takes on the value 1 at least twice.

Note that f has zeros at $x = 1$ and $x = 2$. It also has vertical asymptotes at $x = 0$ and $x = 4$. The right side of the asymptote at $x = 0$ is positive as is the left side of the asymptote at $x = 4$. To apply the IVT, we need to consider an interval on which the function is defined and continuous and at whose endpoints the function takes on values on either side of 1. The interval $[0.1, 1]$ has the required features. Thus f must take on the value 1 somewhere in this interval. Similarly, the interval $[2, 3.99]$ also works. A similar interval to the left of the origin would also work.