Outline

- Characteristic time.
- Solving ODEs approximately using Euler's method and a spreadsheet.
- Solving ODEs using Separation of Variables.

Characteristic time

Consider the ODE

$$\frac{dx}{dt} = -\alpha x.$$

- A solution is $x(t) = e^{-\alpha t}$.
- Shifting gives a bunch more solutions

$$y(t) = e^{-\alpha(t-t_0)} = Ae^{-\alpha t}.$$

- What are the units on α? time^{-1}.
- What is $1/\alpha$? $y(1/\alpha) = Ae^{-\alpha/\alpha} = A/e$.
- So $1/\alpha$ is like half-life but is "1/e"-life instead. Called *mean life* or *characteristic time*.

Characteristic time

Compare this to half-life:

$$y(t_{1/2}) = Ae^{-\alpha t_{1/2}} = \frac{A}{2}.$$

$$e^{-\alpha t_{1/2}} = \frac{1}{2}.$$

$$-\alpha t_{1/2} = \log\frac{1}{2} = -\log 2.$$

$$t_{1/2} = \frac{1}{\alpha}\log 2.$$

So characteristic time is only a factor of $\log 2$ different from half-life.

Characteristic time

What if your equation is not linear? Say $\frac{dx}{dt} = f(x)$ with a steady state at x_{ss}. How "long" does it take to get to the steady state?

Linearize the equation near the steady state:

$$\frac{dx}{dt} = f(x) \approx f(x_{ss}) + f'(x_{ss})(x - x_{ss}) = f'(x_{ss})(x - x_{ss}).$$

Cleaner approximate equation:

$$\frac{dx}{dt} = a(x - b).$$

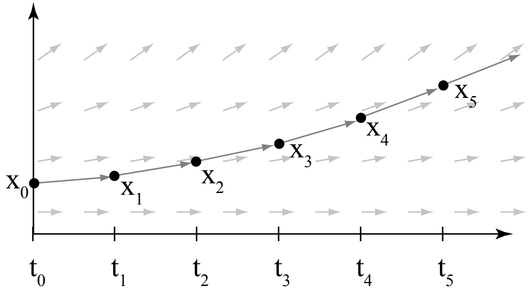
Solution: $x(t) = Ae^{at} + b$.
Steady state: $x_{ss} = b$.
Stability: stable if $a < 0$.
Characteristic time: $\tau = -\frac{1}{a} = -\frac{1}{f'(x_{ss})}$.

Solving an ODE approximately - Euler's method

Consider the ODE

$$\frac{dx}{dt} = f(x, t).$$

- To solve an ODE exactly, follow a slope field continuously.
- To solve an ODE "numerically", follow a slope field discretely.

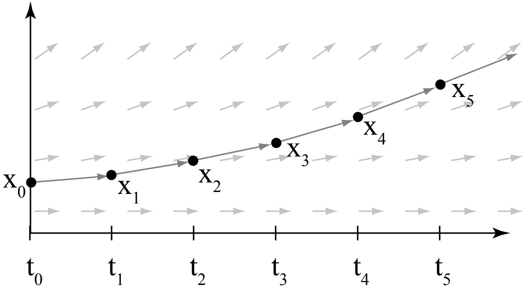


Solving an ODE approximately - Euler's method

- Starting at (t_0, x_0), use the slope field at that point to linearly approximate the solution so that

$$x_1 = x_0 + f(x_0, t_0)(t_1 - t_0).$$

$$x_2 = x_1 + f(x_1, t_1)(t_2 - t_1).$$



Now, to the spreadsheet...

How not to solve a differential equation

- $\dfrac{df}{dx}$ is NOT A FRACTION (even in Physics class).
- What is Jess really doing when he changes this:

$$\frac{dx}{dt} = -kx$$

into this:

$$\int \frac{1}{x}\ dx = -\int k\ dt\ ?$$

Derivatives of inverse functions

- Example: $g(x) = \log x$ is the inverse function of $f(x) = e^x$.
- This means $f(g(x)) = ??x$.
- Suppose we know the derivative of $f(x)$.

$$\frac{d}{dx} f(x) = e^x.$$

- What is the derivative of $h(x) = f(g(x)) = e^{g(x)}$?

$$\frac{d}{dx} h(x) = \frac{d}{dg} f(g(x)) \frac{d}{dx} g(x) = e^{g(x)} g'(x)$$

- But $h(x) = \mathrm{x}$ so $\dfrac{d}{dx} h(x) = 1$.
- So $g'(x) = 1/\left(\dfrac{d}{dg} f(g(x))\right) = 1/e^{g(x)} = 1/x$.

Derivatives of inverse functions

- In general, if $f(g(x)) = x$ then $g'(x) = \dfrac{1}{\frac{d}{dg} f(g(x))}$.
- Try using this to find the derivative of $\arcsin x$.

Solving differential equations

- Want to solve $\frac{d}{dt}x(t) = f(x(t))$.
- Equivalent to $\frac{1}{f(x(t))} \cdot \frac{d}{dt}x(t) = 1$ provided ??$f(x(t)) \neq 0$.
- Suppose we can find a function F such that $\frac{d}{dx}F(x) = \frac{1}{f(x)}$.
- Then $\frac{d}{dt}F(x(t)) = \frac{d}{dx}F(x(t)) \cdot \frac{d}{dt}x(t) = 1$.
- That means $F(x(t)) = t + C$ and we have only to solve this for $x(t)$.

Solving differential equations

Example: $\dfrac{d}{dt}x(t) = -k\ x(t)$

$$\Rightarrow \frac{1}{x(t)}\frac{d}{dt}x(t) = -k$$

$$\Rightarrow \frac{d}{dt}\ln|x(t)| = -k$$

$$\Rightarrow \ln|x(t)| = -kt + C$$

$$\Rightarrow |x(t)| = e^{-kt+C} = e^{C}e^{-kt}$$

$$\Rightarrow x(t) = \pm e^{C}e^{-kt} = De^{-kt}.$$

Try $\dfrac{d}{dt}x(t) = -k\ x(t)^2$.