Outline

- Characteristic time.
- Solving ODEs approximately using Euler's method and a spreadsheet.
- Solving ODEs using Separation of Variables.

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- What is $1/\alpha$? $y(1/\alpha) = Ae^{-\alpha/\alpha} = A/e$.
- So $1/\alpha$ is like half-life but is "1/e"-life instead. Called *mean life* or *characteristic time*.

Characteristic time

Compare this to half-life:

$$y(t_{1/2}) = Ae^{-\alpha t_{1/2}} = \frac{A}{2}.$$

$$e^{-\alpha t_{1/2}} = \frac{1}{2}.$$

$$-\alpha t_{1/2} = \log\frac{1}{2} = -\log 2.$$

$$t_{1/2} = \frac{1}{\alpha}\log 2.$$

So characteristic time is only a factor of $\log 2$ different from half-life.

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Stability:

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Characteristic time: $\tau = -\frac{1}{a} = -\frac{1}{f'(x_{ss})}$.

Solving an ODE approximately - Euler's method

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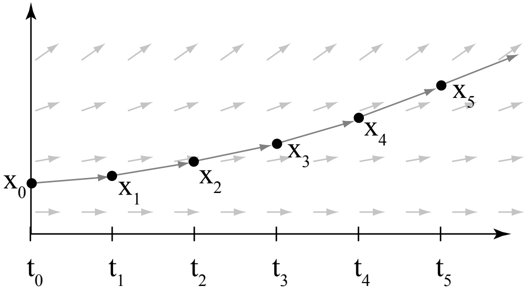
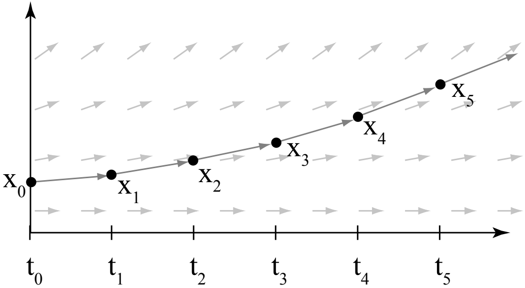


Solving an ODE approximately - Euler's method

- Starting at (t_0, x_0), use the slope field at that point to linearly approximate the solution so that

$$x_1 = x_0 + f(x_0, t_0)(t_1 - t_0).$$

$$x_2 = x_1 + f(x_1, t_1)(t_2 - t_1).$$



Now, to the spreadsheet...

How not to solve a differential equation

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- $\dfrac{df}{dx}$ is NOT A FRACTION (even in Physics class).
- What is Jess really doing when he changes this:

$$\frac{dx}{dt} = -kx$$

into this:

$$\int \frac{1}{x}\ dx = -\int k\ dt\ ?$$

Derivatives of inverse functions

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- In general, if $f(g(x)) = x$ then $g'(x) = \dfrac{1}{\frac{d}{dg} f(g(x))}$.
- Try using this to find the derivative of $\arcsin x$.

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- Then $\frac{d}{dt}F(x(t)) = \frac{d}{dx}F(x(t)) \cdot \frac{d}{dt}x(t) = 1$.
- That means $F(x(t)) = t + C$ and we have only to solve this for $x(t)$.

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Try $\frac{d}{dt}x(t) = -k\ x(t)^2$.