Guide to Integration

Mathematics 101

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February 16, 2007

Table of integrals

Recognise these from a table of derivatives.

The very basics

1. $$\int 1 \, \mathrm{d}x = x + c$$
2. $$\int \frac{1}{x} \, \mathrm{d}x = \log|x| + c$$
3. $$\int x^n \, \mathrm{d}x = \frac{1}{n+1} x^{n+1} + c$$
4. $$\int e^{ax} \, \mathrm{d}x = \frac{1}{a} e^{ax} + c$$

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The very basics

1. $\displaystyle\int 1\,\mathrm{d}x = x + c$
2. $\displaystyle\int \frac{1}{x}\,\mathrm{d}x = \log|x| + c$ — don't forget the $|.|$.
3. $\displaystyle\int x^n\,\mathrm{d}x = \frac{1}{n+1}x^{n+1} + c$ — if $n \neq -1$.
4. $\displaystyle\int e^{ax}\,\mathrm{d}x = \frac{1}{a}e^{ax} + c$

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Trigonometry

1. $$\int \sin(ax)\,dx = \frac{-1}{a}\cos(ax) + c$$
2. $$\int \cos(ax)\,dx = \frac{1}{a}\sin(ax) + c$$
3. $$\int \sec^2(ax)\,dx = \frac{1}{a}\tan(ax) + c$$
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Inverse trig

1. $$\int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}(x/a) + c$$
2. $$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}(x/a) + c.$$

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Inverse trig

1. $$\int \frac{1}{\sqrt{a^2 - x^2}}\,\mathrm{d}x = \sin^{-1}(x/a) + c$$ — need $a > 0$.
2. $$\int \frac{1}{a^2 + x^2}\,\mathrm{d}x = \frac{1}{a}\tan^{-1}(x/a) + c.$$

Substitution rule

From the chain rule we get

$$\int f'(g(x))g'(x)\,\mathrm{d}x = \int f'(u)\,\mathrm{d}u \qquad u = g(x)$$

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$$= f(u) + c = f(g(x)) + c$$

Look for a function and its derivative in the integrand.

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Example $\displaystyle\int \frac{\sin(3\log x)}{x}\,\mathrm{d}x$

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WARNING — you must turn all the x's into the new variable.

Substitution with definite integrals

You have 2 choices of what to do with the integration terminals.

Transform terminals

We make $u = \log x$ — so change the terminals too.

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&= \left[\frac{-1}{3}\cos(3u)\right]_{\log 1 = 0}^{\log 2} \\
&= \frac{-1}{3}\cos(3\log 2) + \frac{1}{3}\cos(0) \\
&= \frac{-1}{3}\cos(3\log 2) + \frac{1}{3}
\end{aligned}$$

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You have 2 choices of what to do with the integration terminals.

Keep terminals, remember to change everything back to x

$$\int_1^2 \frac{\sin(3\log x)}{x}\,\mathrm{d}x = \int_{x=1}^{x=2} \sin 3u\,\mathrm{d}u$$

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Of course the answers are the same.

Trigonometric integrals

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Useful trig identities

$$\begin{aligned} \cos^2 x + \sin^2 x &= 1 \\ 1 + \tan^2 x &= \sec^2 x \\ \cos^2 x &= \frac{1}{2}(1 + \cos 2x) \\ \sin^2 x &= \frac{1}{2}(1 - \cos 2x) \end{aligned}$$

Trigonometric integrals

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- If a and b are both even then use

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- If a or b is odd then use

$$\cos^2 x = 1 - \sin^2 x$$
$$\sin^2 x = 1 - \cos^2 x$$

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Now this is not at all obvious, but you should see it...

$$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx$$

Now set $u = \sec x + \tan x$:

$$\frac{du}{dx} = \sec x \tan x + \sec^2 x = \sec x \left(\tan x + \sec x \right)$$

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Now set $u = \sec x + \tan x$:

$$\frac{\mathrm{d}u}{\mathrm{d}x} = \sec x \tan x + \sec^2 x = \sec x \left(\tan x + \sec x \right)$$

Hence we have

$$\int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \mathrm{d}x = \int \frac{1}{u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x$$

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Hence we have

$$\begin{aligned} \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) dx &= \int \frac{1}{u} \cdot \frac{du}{dx} \, dx \\ &= \int \frac{1}{u} \, du = \log |u| + c \end{aligned}$$

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$$\begin{aligned} \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x} \right) \mathrm{d}x &= \int \frac{1}{u} \cdot \frac{\mathrm{d}u}{\mathrm{d}x} \, \mathrm{d}x \\ &= \int \frac{1}{u} \, \mathrm{d}u = \log |u| + c \\ &= \log |\sec x + \tan x| + c \end{aligned}$$

Integration by parts

From the product rule we get

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$$

- Frequently used when you have the product of 2 different types of functions.

Integration by parts

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$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$$

- Frequently used when you have the product of 2 different types of functions.
- You have to choose $f(x)$ and $g'(x)$ — there are 2 options.
- Usually one will work and the other will not.

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$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$$

$$\int xe^x\,\mathrm{d}x = xe^x - \int e^x \cdot 1\,\mathrm{d}x$$

$$= xe^x - e^x + c$$

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  $f = e^x$ and $g' = x$ — so $f' = e^x$ and $g = x^2/2$

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$$\int xe^x \, \mathrm{d}x = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x \, \mathrm{d}x$$

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$$\int xe^x \, \mathrm{d}x = \frac{x^2}{2} e^x - \int \frac{x^2}{2} e^x \, \mathrm{d}x$$

- This is not getting easier, so stop!

Sometimes one of the parts is “1”.

Example $\int \log x \, \mathrm{d}x$

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Example $\int \log x \, dx$

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- Choose $f = \log x$ and $g' = 1$ — so $f' = 1/x$ and $g = x$:

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- Choose $f = \log x$ and $g' = 1$ — so $f' = 1/x$ and $g = x$:

$$\int f(x)g'(x)\,\mathrm{d}x = f(x)g(x) - \int g(x)f'(x)\,\mathrm{d}x$$

$$\int \log x\,\mathrm{d}x = x\log x - \int x/x\,\mathrm{d}x$$

$$= x\log x - x + c$$

Trigonometric substitutions

Based on

$$\sin^2\theta = 1 - \cos^2\theta$$
$$\tan^2\theta + 1 = \sec^2\theta$$

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Things to associate

If the integrand contains

$$\sqrt{a^2 - x^2} \longrightarrow \qquad \sin^2\theta = 1 - \cos^2\theta$$
$$a^2 + x^2 \longrightarrow \qquad 1 + \tan^2\theta = \sec^2\theta$$

Trigonometric substitutions

Compute $\int (5 - x^2)^{-3/2}\,\mathrm{d}x$

- Contains $\sqrt{a^2 - x^2}$ so put $x = \sqrt{5}\sin\theta$.

Trigonometric substitutions

Compute $\int (5-x^2)^{-3/2}\,\mathrm{d}x$

- Contains $\sqrt{a^2-x^2}$ so put $x=\sqrt{5}\sin\theta$.
- Hence $\frac{\mathrm{d}x}{\mathrm{d}\theta}=\sqrt{5}\cos\theta$ and

$$\int (5-x^2)^{-3/2}\,\mathrm{d}x=\int \frac{\sqrt{5}\cos\theta}{5^{3/2}(1-\sin^2\theta)^{3/2}}\,\mathrm{d}\theta$$

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- Hence $\frac{\mathrm{d}x}{\mathrm{d}\theta} = \sqrt{5}\cos\theta$ and

$$\begin{aligned}\int (5 - x^2)^{-3/2}\,\mathrm{d}x &= \int \frac{\sqrt{5}\cos\theta}{5^{3/2}(1 - \sin^2\theta)^{3/2}}\,\mathrm{d}\theta \\ &= \int \frac{\cos\theta}{5(\cos^2\theta)^{3/2}}\,\mathrm{d}\theta\end{aligned}$$

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&= \int \frac{\cos\theta}{5(\cos^2\theta)^{3/2}}\,\mathrm{d}\theta \\
&= \int \frac{1}{5\cos^2}\,\mathrm{d}\theta = \frac{1}{5}\int \sec^2\theta\,\mathrm{d}\theta
\end{aligned}$$

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\int (5 - x^2)^{-3/2}\, \mathrm{d}x &= \int \frac{\sqrt{5}\cos\theta}{5^{3/2}(1 - \sin^2\theta)^{3/2}}\, \mathrm{d}\theta \\
&= \int \frac{\cos\theta}{5(\cos^2\theta)^{3/2}}\, \mathrm{d}\theta \\
&= \int \frac{1}{5\cos^2}\, \mathrm{d}\theta = \frac{1}{5}\int \sec^2\theta\, \mathrm{d}\theta \\
&= \frac{1}{5}\tan\theta + c
\end{aligned}$$

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- We aren't done yet — we have to change back to the x variable.

Trigonometric substitutions

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- We substituted $x = \sqrt{5}\sin\theta$ and got $\frac{1}{5}\tan\theta + c$
- We can express $\tan\theta$ in terms of $\sin\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1 - \sin^2\theta}}$$

$$= \frac{x/\sqrt{5}}{\sqrt{1 - x^2/5}} = \frac{x}{\sqrt{5}\sqrt{1 - x^2/5}} = \frac{x}{\sqrt{5 - x^2}}$$

Trigonometric substitutions

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- We substituted $x = \sqrt{5}\sin\theta$ and got $\frac{1}{5}\tan\theta + c$
- We can express $\tan\theta$ in terms of $\sin\theta$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$$
$$= \frac{x/\sqrt{5}}{\sqrt{1-x^2/5}} = \frac{x}{\sqrt{5}\sqrt{1-x^2/5}} = \frac{x}{\sqrt{5-x^2}}$$

- Hence the integral is

$$\int (5-x^2)^{-3/2}\,dx = \frac{x}{5\sqrt{5-x^2}} + c$$

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Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

- Contains $a^2 + x^2$, so sub $x = 2\tan\theta$.
- Hence $\frac{\mathrm{d}x}{\mathrm{d}\theta} = 2\sec^2\theta$ and

$$\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \int \frac{2\sec^2\theta}{\sqrt{4+4\tan^2\theta}}\,\mathrm{d}\theta$$

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Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

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&= \int \frac{2\sec^2\theta}{2\sqrt{1+\tan^2\theta}}\,\mathrm{d}\theta = \int \frac{\sec^2\theta}{\sqrt{\sec^2\theta}}\,\mathrm{d}\theta \\
&= \int \sec\theta\,\mathrm{d}\theta
\end{aligned}$$

Compute $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

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- We have assumed $\sec\theta > 0$. We did similarly in the previous example.

Trigonometric substitutions

Compute $\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x$

- We substituted $x = 2\tan\theta$ and got

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Trigonometric substitutions

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- Hence: $\displaystyle\int \frac{1}{\sqrt{4+x^2}}\,\mathrm{d}x = \log|\sqrt{1+x^2/4} + x/2| + c.$

Trigonometric substitutions

- Sometimes you need to complete the square in order to get started.

Try $\frac{dx}{4x^2 + 12x + 13}$

Partial fractions

Based on partial fraction decomposition of rational functions

- There are some very general rules for this technique.
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- Any polynomial with real coefficients can be factored into linear and quadratic factors with real coefficients

$$
\begin{aligned}
Q(x) = {} & k(x-a_1)^{m_1}(x-a_2)^{m_2}\cdots(x-a_j)^{m_j} \\
& \times (x^2+b_1x+c_1)^{n_1}(x^2+b_2x+c_2)^{n_2}\cdots(x^2+b_lx+c_l)^{n_l}
\end{aligned}
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Partial fractions

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$$
\begin{aligned}
f(x) = {} & \frac{A_{11}}{(x-a_1)^1} + \frac{A_{12}}{(x-a_1)^2} + \cdots + \frac{A_{1m_1}}{(x-a_1)^{m_1}} \\
& + \text{similar terms for each linear factor} \\
& + \frac{B_{11}x + C_{11}}{(x^2+b_1x+c_1)^1} + \frac{B_{12}x + C_{12}}{(x^2+b_1x+c_1)^2} + \cdots \frac{B_{1n_1}x + C_{1n_1}}{(x^2+b_1x+c_1)^{n_1}} \\
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- Once in this form, we can integrate term-by-term.

Partial fractions

$$\int \frac{dx}{x(x-1)}$$

- Write in partial fraction form

$$\frac{1}{x(x-1)} = \frac{A}{x} + \frac{B}{x-1}$$

Now find A and B.

$$= \frac{A(x-1)+Bx}{x(x-1)}$$

Compare numerators

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- Hence we have 2 equations

$$\left.\begin{aligned} A+B &= 0 \\ -A &= 1 \end{aligned}\right\} \Rightarrow A = -1, B = 1$$

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$$\begin{aligned}\int \frac{1}{x(x-1)}\,dx &= -\int \frac{1}{x}\,dx + \int \frac{1}{x-1}\,dx \\ &= -\log|x| + \log|x-1| + c \\ &= \log\left|\frac{x-1}{x}\right| + c\end{aligned}$$

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Try $\int \frac{1}{x^2-a^2}\,\mathrm{d}x$.

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About 100 Integrals...

25. $\int \tan^3 x \sec^4 x \, dx$

27. $\int \tan x \sec^5 x \, dx$

29. $\int \cot^3 x \csc^2 x \, dx$

31. $\int_{\pi/4}^{\pi/2} \cot^3 x \csc^3 x \, dx$

33. $\int \cot x \csc^{-2} x \, dx$

35. $\int \frac{\tan x}{\cos^3 x} \, dx$

37. $\int \frac{\tan^2 x}{\sec^5 x} \, dx$

26. $\int_0^{\pi/3} \tan x \sec^{3/2}$

28. $\int \csc^3 x \, dx$

30. $\int \cot^3 s \csc^4 s \, ds$

32. $\int_{\pi/3}^{\pi/4} \cot x \csc^3 x$.

34. $\int \frac{\cot t}{\csc^3 t} \, dt$

36. $\int \tan^3 x \csc^2 x \, d.$

38. $\int \sin^5 w \cot^3 w$

39. $\int \frac{\tan x}{\sec^2 x} \, dx$

40. $\int \frac{\tan^5 x}{\sec^4 x} \, dx$

41. $\int \tan^2 x \, dx$ (*Hint:* $\tan^2 x = \sec^2 x - 1$.)

42. $\int \tan^3 x \, dx$

43. $\int \tan^4 x \, dx$

44. $\int \cot^5 x \, dx$

45. $\int \sin 2x \cos 3x \, dx$

46. $\int_0^{2\pi/3} \sin x \cos 2x \, dx$

47. $\int \sin(-4x) \cos(-$

48. $\int \sin 3x \cos \frac{1}{2}x \, dx$

49. $\int \sin \frac{1}{2}x \cos \frac{2}{3}x \, dx$

2. $\int x \sec^2 x \, dx$

4. $\int x \ln x^2 \, dx$

6. $\int x^2 \ln x \, dx$

8. $\int xe^{-x} \, dx$

10. $\int x^2 \sin x \, dx$

12. $\int e^x \sin x \, dx$

14. $\int \frac{\sin x}{e^x} \, dx$

16. $\int t \cdot 3^{-t} \, dt$

18. $\int \log_6 x \, dx$

20. $\int t^2 \cosh t \, dt$

22. $\int \sin^{-1} 3x \, dx$

33. $\int \frac{1}{(9x^2 - 4)^{5/2}} \, dx$

35. $\int \sqrt{4 + x^2} \, dx$

37. $\int_{3\sqrt{2}}^{6} \frac{1}{x^4 \sqrt{x^2 - 9}} \, dx$

34. $\int \frac{\sqrt{4 + x^2}}{x^3} \, dx$

36. $\int \frac{\sqrt{x^2 + 1}}{x^4} \, dx$

38. $\int \frac{2x - 3}{\sqrt{4x - x^2 - 3}} \, dx$

1. $\int \frac{x}{x + 1} \, dx$

3. $\int \frac{x^2}{x^2 - 1} \, dx$

2. $\int \frac{x^2}{x^2 + 1} \, dx$

4. $\int \frac{t^2 - 1}{t^2 + 1} \, dt$

5. $\int \frac{x^2 + 4}{x(x - 1)^2} \, dx$

7. $\int_3^4 \frac{5}{(x - 2)(x + 3)} \, dx$

9. $\int \frac{3t}{t^2 - 8t + 15} \, dt$

6. $\int \frac{2x^3 + x^2 + 12}{x^2 - 4} \, dx$

8. $\int \frac{5x}{(x - 2)(x + 3)} \, dx$

10. $\int \frac{2}{x^2 - x - 6} \, dx$

15. $\int_0^{5/4} \frac{1}{\sqrt{25 - 4x^2}} \, dx$

17. $\int \frac{1}{\sqrt{4x^2 + 4x + 2}} \, dx$

19. $\int \frac{1}{(1 - 2w^2)^{5/2}} \, dw$

21. $\int_0^1 \frac{1}{2x^2 - 2x + 1} \, dx$

23. $\int \sqrt{x - x^2} \, dx$

25. $\int \frac{x^2}{\sqrt{9x^2 - 1}} \, dx$

27. $\int \frac{1}{x\sqrt{x^2 + 4}} \, dx$

29. $\int \frac{1}{x^2\sqrt{4x^2 - 9}} \, dx$

31. $\int \frac{x^2}{\sqrt{1 + x^2}} \, dx$

1. $\int_0^{1/2} \sqrt{1 - 4x^2} \, dx$

3. $\int_{-2}^{2} \sqrt{1 - \frac{x^2}{4}} \, dx$

5. $\int \frac{1}{(3 - x^2)^{3/2}} \, dx$

7. $\int \frac{1}{(x^2 + 1)^{3/2}} \, dx$

9. $\int_0^1 \frac{1}{(3x^2 + 2)^{5/2}} \, dx$

11. $\int_2^{2\sqrt{2}} \frac{\sqrt{x^2 - 4}}{x} \, dx$

13. $\int \frac{1}{(9 + t^2)^2} \, dt$

2. $\int_0^4 \sqrt{16 - x^2} \, dx$

4. $\int \sqrt{1 + (2x - 1)^2} \, dx$

6. $\int_{4\sqrt{2}}^{8} \frac{1}{(t^2 - 16)^{3/2}} \, dt$

8. $\int \frac{1}{(2x^2 + 1)^{3/2}} \, dx$

10. $\int \frac{x^2}{(x^2 - 2)^{5/2}} \, dx$

12. $\int \frac{1}{(y^2 - 4)^{3/2}} \, dy$

14. $\int_1^{\sqrt{2}} \frac{1}{\sqrt{2x^2 - 1}} \, dx$

39. $\int \frac{x}{\sqrt{2x^2 + 12x + 19}} \, dx$

40. $\int \frac{1}{(w^2 + 2w + 5)^{3/2}} \, dw$

42. $\int \frac{e^{3w}}{\sqrt{1 - e^{2w}}} \, dw$

*41. $\int \sqrt{x^2 + 6x + 5} \, dx$

43. $\int e^w \sqrt{1 + e^{2w}} \, dw$

1. $\int \sqrt{4x - 5} \, dx$; $u = 4x - 5$

2. $\int (1 - 5x^2)^{2/3}(10x) \, dx$; $u = 1 - 5x^2$

3. $\int \cos \pi x \, dx$; $u = \pi x$

4. $\int 3 \sin(-2x) \, dx$; $u = -2x$

5. $\int x \cos x^2 \, dx$; $u = x^2$

6. $\int \sin^3 t \cos t \, dt$; $u = \sin t$

7. $\int \cos^{-4} t \sin t \, dt$; $u = \cos t$

8. $\int \frac{2t - 3}{(t^2 - 3t + 1)^2} \, dt$; $u = t^2 - 3t + 1$

9. $\int \frac{2t - 3}{(t^2 - 3t + 1)^{7/2}} \, dt$; $u = t^2 - 3t + 1$

10. $\int x\sqrt{x - 1} \, dx$; $v = x - 1$

11. $\int (x - 1)\sqrt{x + 1} \, dx$; $v = x + 1$

12. $\int x^2\sqrt{x + 3} \, dx$; $v = x + 3$