## Science One Mathematics Exam <br> 17 December 2009 <br> Answer key

1. Determine whether each statement is true or false. Circle T or F as appropriate. Give a few words of justification for each statement or a counter-example where applicable.
(a) Every function that is continuous is also differentiable.

Ans:
False. $f(x)=|x|$ is continuous everywhere but not differentiable at $x=0$.
(b) If $f^{\prime}(-1)>0$ and $f^{\prime}(1)<0$ then $f$ has a horizontal tangent line at some $c$ in the interval $(-1,1)$. T F Ans:
False. If we were told that $f$ is differentiable so that $f^{\prime}$ is continuous on the interval $[-1,1]$, the Intermediate Value Theorem would ensure the existence of a horizontal tangent line. But we are not told that here. A counter-example would be $g(x)=-|x|$.
(c) A solution to the equation $d x / d t=g(x)$ cannot have a local maximum.

## Ans:

True. If $x(t)$ could have a local maximum, then there would be two points on the graph of $x(t)$, say at $t_{1}$ and $t_{2}$, with the same height, $x\left(t_{1}\right)=x\left(t_{2}\right)$, but with slopes of opposite sign. But as $x(t)$ satisfies the differential equation, it tells us that $x^{\prime}\left(t_{1}\right)=g\left(x\left(t_{1}\right)\right)$ and that $x^{\prime}\left(t_{2}\right)=g\left(x\left(t_{2}\right)\right)$. Because $g$ is a function (implied but not explicitly stated in the question), we conclude that $x^{\prime}\left(t_{1}\right)=x^{\prime}\left(t_{2}\right)$. This contradicts the conclusion above that the slopes are of opposite sign so we must have been wrong to assume a maximum was possible.
(d) Mathematics is way cooler than physics, chemistry and biology. Think carefully.

T F
Ans:
We were just fishing for entertaining answers here. Marking exams can get boring. The best ones were:

- True since the entropy of physics, chemistry and biology are greater than the entropy of math:

$$
\frac{d \sigma_{m a t h}}{d U}>\frac{d \sigma_{p c b}}{d U} \Rightarrow \beta_{m a t h}>\beta_{p c b} \Rightarrow T_{m a t h}<T_{p c b}
$$

- True. (Pure) Mathematics is immune to the ugliness of the world.
- True. I have discovered a truly marvelous proof of this but this space is ...
- True. I want to pass.
- True. If evaluated alone. Tick, tock, tick, tock. AHHHHH! Tick, tock, tick, tock. (silence)

2. Suppose that $f(1)=f^{\prime}(1)=1$. Define $g(x)=f\left(x^{3}\right)$. Use a linear approximation to the function $g$ (not a linear approximation to the function $f$ ) to estimate $g(1.1)$.

## Ans:

First, I need to calculate $g^{\prime}(x)$ using the chain rule:

$$
g^{\prime}(x)=f^{\prime}\left(x^{3}\right) \cdot 3 x^{2}
$$

Next, I use this to write down the linear approximation to $g$ near $x=1$ :

$$
g(x) \approx g(1)+g^{\prime}(1)(x-1)=f\left(1^{3}\right)+f^{\prime}\left(1^{3}\right) 31^{2}(x-1)=1+3(x-1)=3 x-2
$$

To put that concisely,

$$
g(x) \approx 3 x-2
$$

3. (a) Carefully state the definition of the derivative of a function $f$ at a point $a$.

Ans:
The derivative of the function $f$ at a point $a$ is defined as

$$
f^{\prime}(a)=\lim _{x \rightarrow a} \frac{f(x)-f(a)}{x-a}
$$

if this limit exists.
(b) Use the definition of the derivative to compute $f^{\prime}(1)$ where $f(x)=x /(1+x)$. No credit for any other method of calculation.
Ans:
You should be able to figure this one out. The key step, as usual with these, is to rearrange things so that there is a common factor $(x-1)$ on top and bottom that you can cancel. Putting the fraction over a common denominator will help.
4. Let $f(x)=x e^{-x^{2} / 2}$.
(a) Identify intervals where $f$ is increasing, decreasing, concave up, and concave down.
(b) Find all vertical and horizontal asymptotes.
(c) Sketch the graph of $y=f(x)$.

## Ans:

You should find (after calculating appropriate derivatives) that the function is increasing between -1 and 1 and decreasing elsewhere and that it is concave up between $-\sqrt{3}$ and 0 and again above $\sqrt{3}$ and concave down elsewhere. It has no vertical asymptotes and the $x$-axis is a horizontal asymptote at both $\infty$ and $-\infty$. Your sketch should like like this:

5. Assume that the force a kinesin motor can generate when moving at velocity $v$ is given by

$$
F_{k i n}(v)=F_{k i n}^{0}\left(1-\frac{v}{v_{k i n}}\right)
$$

A laser trap is an experimental setup in which a bead is attached to a kinesin motor and illuminated by a laser beam. Due to refraction and conservation of momentum, the beam exerts a force on the bead much like a Hookean spring, that is, $F_{\text {trap }}(x)=-k x$ where $x$ is the position of the bead as measured from the center of the beam (right is positive, left is negative). Finally, the drag force on the bead is proportional to its velocity, $F_{\text {drag }}(v)=-\gamma v$. At low Reynolds number, the equation of motion is $F_{\text {kin }}(v)+F_{\text {drag }}(v)+F_{\text {trap }}(x)=0$.
(a) Rewrite the equation of motion in the form of a differential equation.

Ans:
The equation written out in full is

$$
F_{k i n}^{0}\left(1-\frac{v}{v_{k i n}}\right)-\gamma v-k x=0
$$

Because $v=d x / d t$, we want to solve for $v$ so that the equation is in the convenient form:

$$
\frac{d x}{d t}=\frac{F_{k i n}^{0}-k x}{\frac{F_{k i n}^{0}}{v_{k i n}}+\gamma}=\frac{k v_{k i n}}{F_{k i n}^{0}+\gamma v_{k i n}}\left(\frac{F_{k i n}^{0}}{k}-x\right)=A(B-x)
$$

where $A=\frac{k v_{k i n}}{F_{k i n}^{0}+\gamma v_{k i n}}$ and $B=\frac{F_{k i n}^{0}}{k}$.
(b) At what position does the bead end up after a long period of time? Is it to the left or right of the center of the bead?
Ans:
Because there is a minus sign in front of the $x$ in the equation, the steady state at $x=B$ is stable. Hence, "after a long period of time", the bead ends up approaching $x=\frac{F_{k i n}^{0}}{k}$. Assuming both these parameters are positive, the bead ends up to the right of the center of the beam.
(c) What is the characteristic time associated with reaching that position? Express your answer in terms of the parameters $\gamma, F_{k i n}^{0}, v_{k i n}$ and $k$.
Ans:
The characteristic time is simply $1 / A$ which is $\frac{F_{k i n}^{0}+\gamma v_{k i n}}{k v_{k i n}}$. Note that "a long period of time" really means "much longer than the characteristic time".
6. Consider the differential equation

$$
\frac{d v}{d t}=v(1-v)(v-0.1)-\beta
$$

which is a dimensionless model for a non-linear resistor in parallel with a capacitor (as well as a simple electrophysiology model). The parameter $\beta>0$ represents a constant current applied across the circuit.
(a) Draw the phase line for this equation when $\beta=0$.

Ans:
Ask me (Eric) if you need help on this one - I want to post this asap and don't have a good sketching tool available right now.
(b) With $\beta=0$, sketch a solution that starts at value of $v(0)=0.11$.

Ans:
Ask me (Eric) if you need help on this one - I want to post this asap and don't have a good sketching tool available right now.
(c) For a large enough applied current $\beta$, there is only one steady state. Describe in words how you would find that value of $\beta$ above which there is only one steady state.
Ans:
For convenience, let's call the right hand side of the equation, when $\beta=0, f(v)$. That is, $f(v)=$ $v(1-v)(v-0.1)$. Increasing $\beta$ shifts all slopes down. As there is a local maximum in $f(v)$ between the steady states 0.1 and 1 for case of $\beta=0$, increasing $\beta$ brings these steady states closer together until, at some value of $\beta$, they "collide", and for values of $\beta$ larger than that, there is only one steady state remaining (the one off to the left). The key value of $\beta$ is the one that is precisely the height of the local maximum of $f(v)$. Thus, solve $f^{\prime}\left(v_{\max }\right)=0$ for $v_{\max }$ - there will be two solutions; we want the maximum on the right, not the minimum on the left so choose the larger solution, i.e. the + in the quadratic formula. Any value of $\beta>f\left(v_{\max }\right)$ will give a phase line with only one steady state.
7. Let $f$ be a continuous, bounded function - that is, there are constants $A$ and $B$ such that $A<f(x)<B$ for all real values of $x$. Define $g(x)=f(x)-x$.
(a) Show that there exist numbers $a$ and $b$ such that $f(a)>a$ and $f(b)<b$.

Ans:
This is equivalent to showing that there is a number $a$ such that $g(a)>0$ and a number $b$ such that $g(b)<0$. We know that $A<f(x)<B$ so if we subtract $x$ from both sides, we can get a similar inequality for $g$. That is, $A-x<f(x)-x<B-x$ which is the same as $A-x<g(x)<B-x$. What values of $x$ will give us the results we want? Try $x=A$. This gives $0<g(A)<B-A$. The first half of that inequality tells us that we found our $a$; that is, $a=A$. Similarly, plugging in $x=B$ shows us that $B$ is a perfectly good candidate for $b$.
(b) Use part (a) to show that $f$ has a fixed point (a number $c$ such that $f(c)=c$ ). (Hint: use the Intermediate Value Theorem and the function $g(x)=f(x)-x$.)
Ans:
As suggested in the hint, we apply the IVT to $g(x)$. We've already shown that $g(A)>0$ and $g(B)<0$. Because $f$ is continuous, $g$ will also be continuous. So the IVT applies and ensures that there is a $c$ between $A$ and $B$ such that $g(c)=0$. This is equivalent to saying that there is a $c$ such that $f(c)=c$.

