Midterm solutions (term 2)

Q1 (a) Take limits of both sides of the recursive definition to get $a^* = a^*/3 + 1$. Solve for a^* to get $\lim_{n\to\infty} a_n = 3/2$.

(b) Show that a_n is monotone increasing (by induction). Then show that a_n is bounded above by $3/2$ (also by induction). Conclude that a_n converges.

Q2 (a) Ignoring the alternating minus sign, the sequence converges to zero. So by the alternating series test, the given series converges.

(b) The sequence being summed can be rewritten as $a_n = (2n)!/(2(n!))^2$. The ratio test requires that we consider the ratio a_{n+1}/a_n. Simplifying this expression, we find it is equal to $\frac{(2n+2)(2n+1)}{(n+1)^2}$ which converges to 4. So the series diverges.

Q3 (a) Check your textbook.

(b) The quantity in parentheses is just $\int_0^x \sin(s)\, ds$. The derivative of this with respect to x is $\sin(x)$ by the FTC.

Q4 (a) $\cos(x) = 1 - x^2/2! + x^4/4! - ... + (-1)^n x^{2n}/(2n)! + ...$ for all values of x.

(b) Because the limit is "as x goes to zero", we only need to keep the smallest power of x in in the Taylor expansion of both numerator and denominator. The numerator is approximately $x^4/2$ for small x and the denominator is approximately $x^4/4$ for small x. The ratio of these two is 2 so the limit requested is 2.

Q5 See the "Bolzano-Weierstrass Theorem" on Wikipedia for a concise proof.