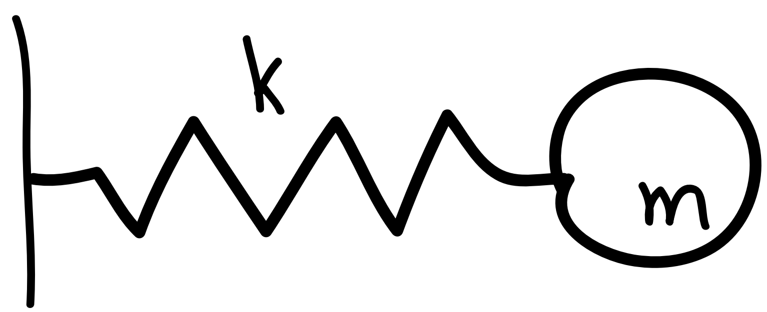
Today

- Mass-springs as models for everything.
- Forced vibrations
    - Newton’s 2nd Law with external forcing.
    - Forced mass-spring system without damping away from resonance.
    - Forced mass-spring system without damping at resonance.
    - Forced mass-spring system with damping.
- Midterm (Feb 2, in class) - Everything up to and including Tuesday Jan 26 (Method of Undetermined Coefficients).

Applications - vibrations

Mass-spring systems



$$E = \frac{1}{2}k(x-x_0)^2$$

$$F = -\frac{dE}{dx} = -k(x-x_0)$$

$$ma = F$$

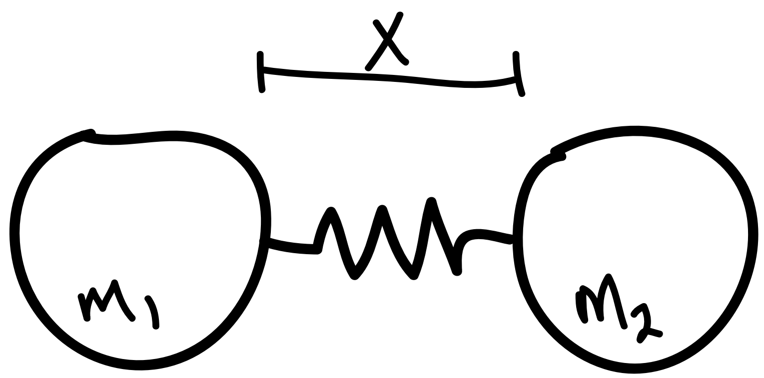
$$ma = -k(x-x_0)$$

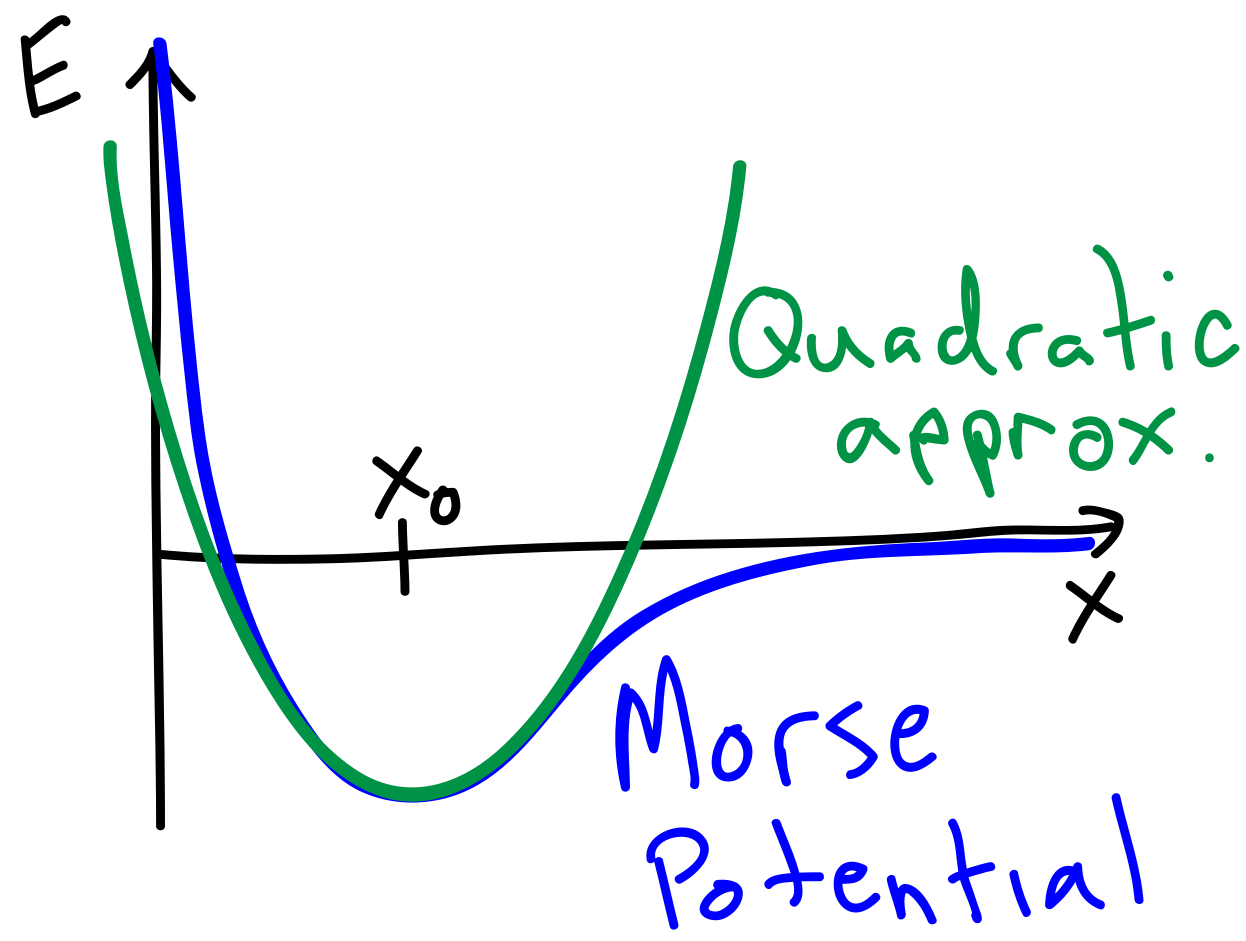
$$mx'' = -k(x-x_0)$$

$$mx'' + kx = kx_0$$

Applications - vibrations

Molecular bonds

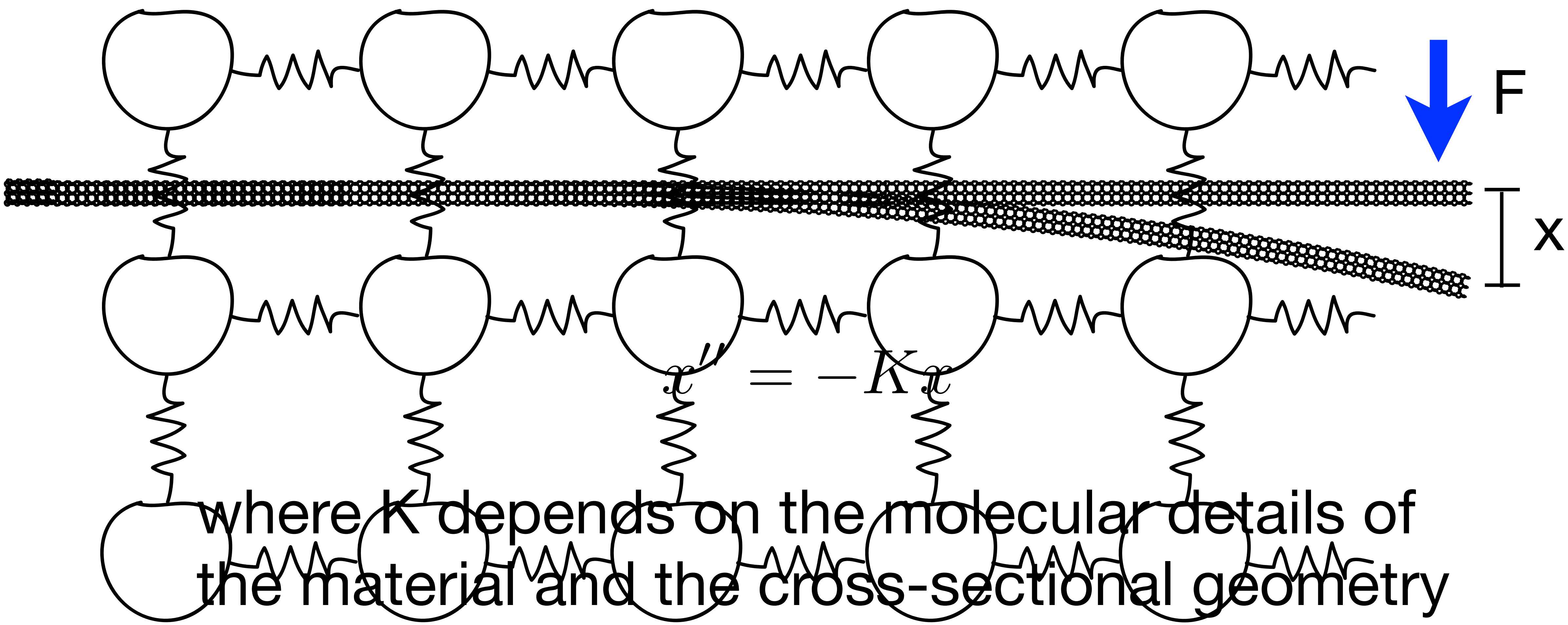




Applications - vibrations

Solid mechanics

e.g. tuning fork, bridges, buildings



$$x'' = -Kx$$

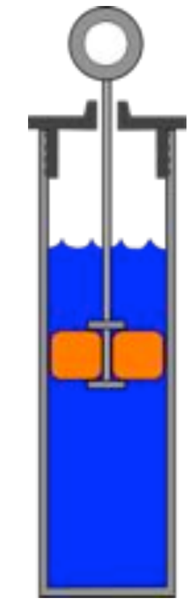
where K depends on the molecular details of the material and the cross-sectional geometry of the rod.

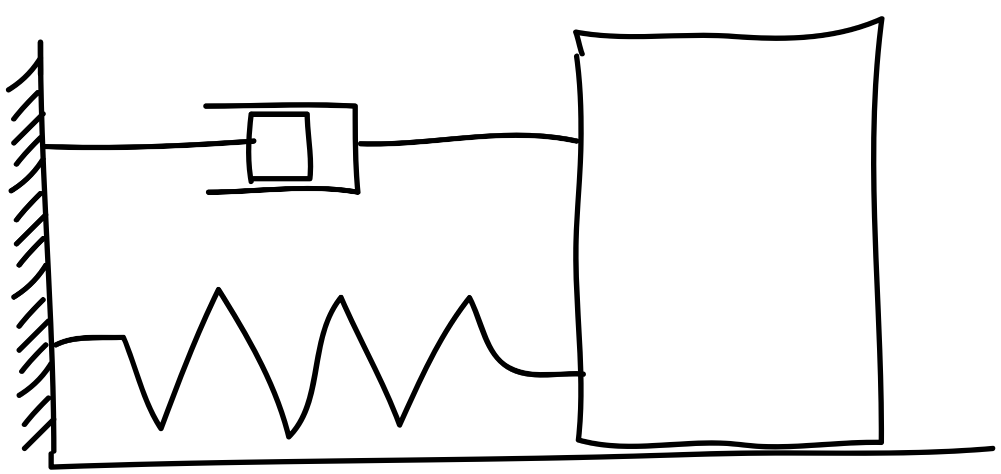
Applications - vibrations

- So far, no x’ term so no exponential decay in the solutions.

- Dashpot - mechanical element that adds friction.

  - sometimes an abstraction that accounts for energy loss.





Kelvin-Voigt model

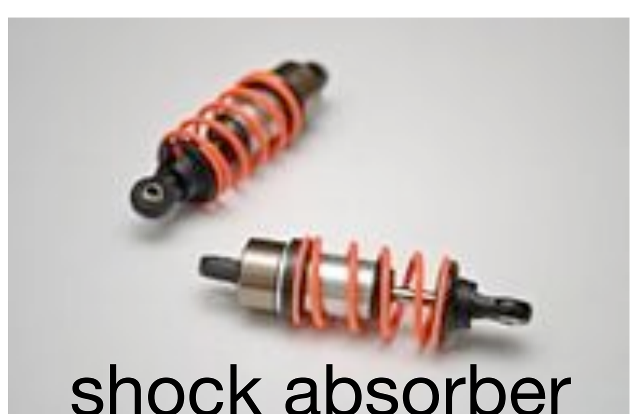
$$ma = -k(x - x_0) - \gamma v$$

$$mx'' = -k(x - x_0) - \gamma x'$$

$$mx'' + \gamma x' + kx = kx_0$$

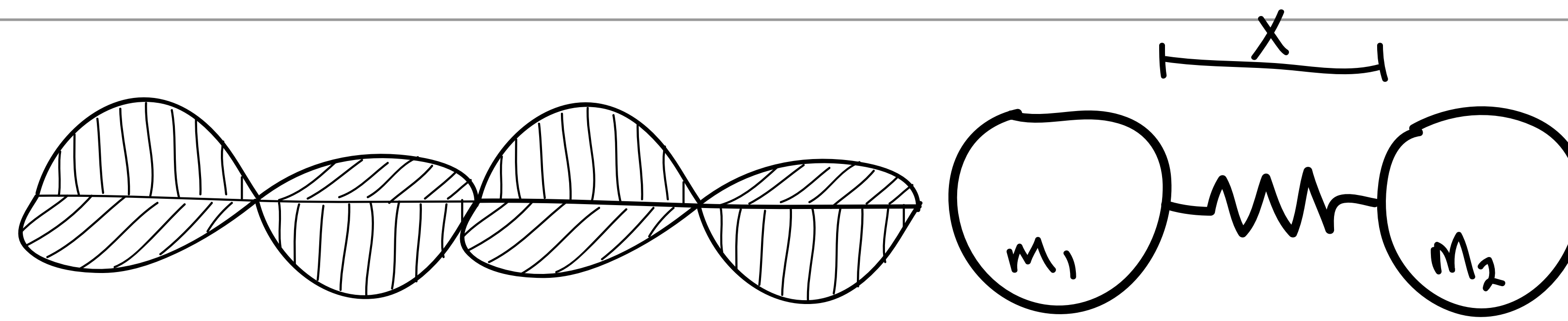
$$y = x - x_0$$

$$my'' + \gamma y' + ky = 0$$

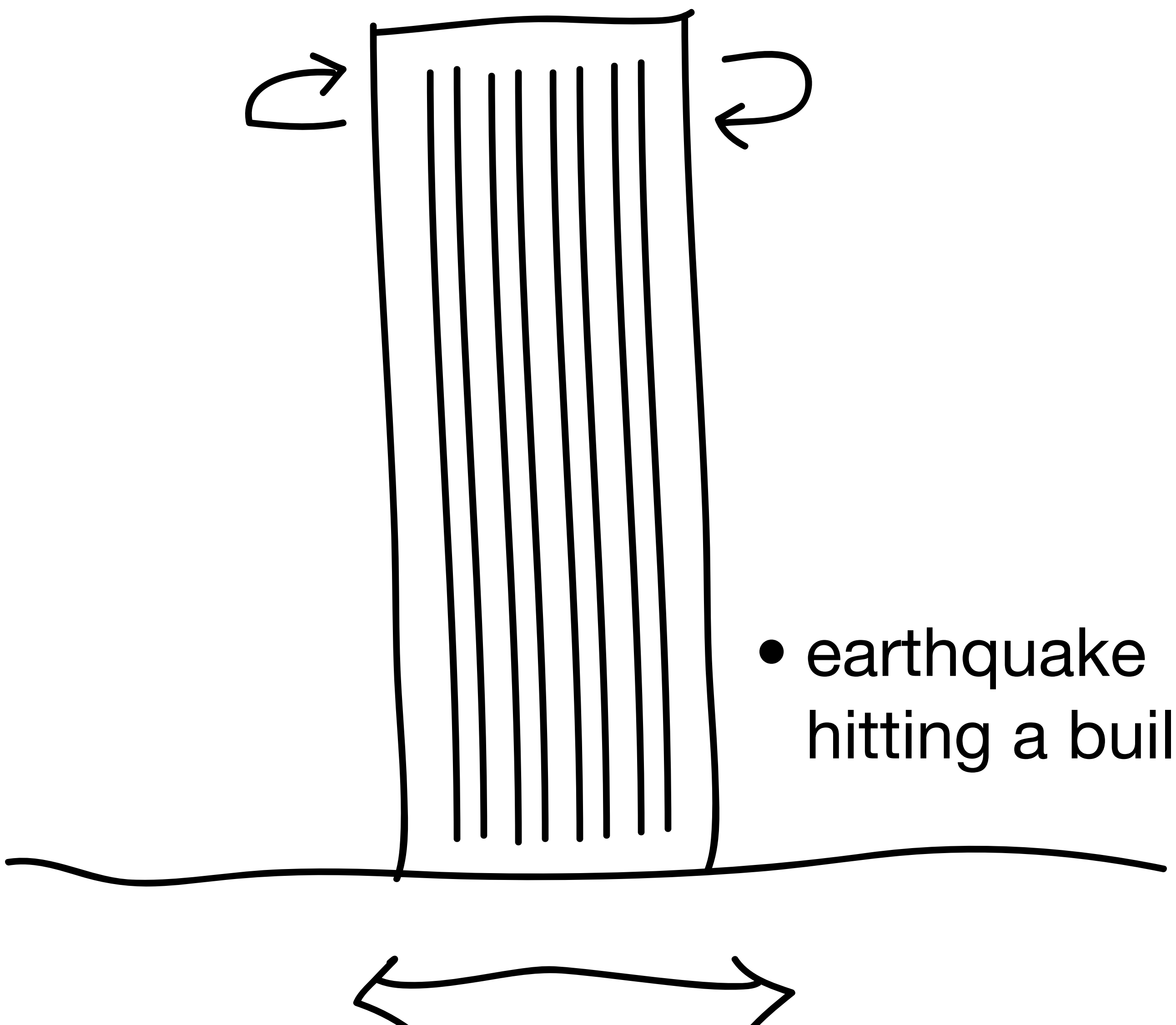


shock absorber

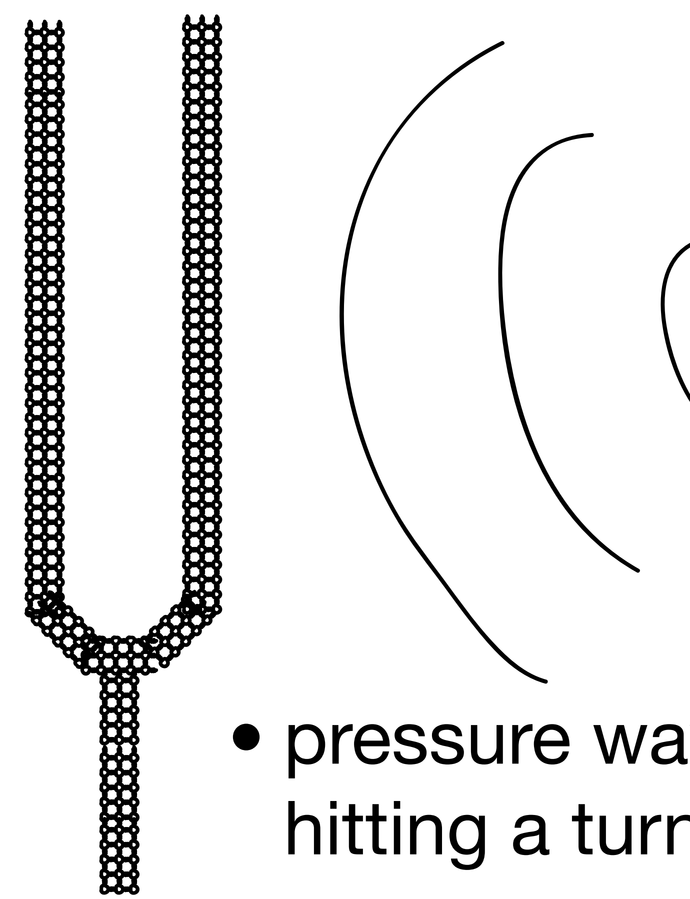
Applications - forced vibrations



- light hitting a molecular bond



- earthquake hitting a building



- pressure waves (sound) hitting a turning fork.

Applications - vibrations, undamped

- Undamped mass spring

$$mx'' + kx = 0$$

(A) $x(t) = C_1 e^{-\omega_0 t} + C_2 e^{\omega_0 t}$

(B) $x(t) = C_1 e^{-\omega_0 t} + C_2 t e^{-\omega_0 t}$

☆ (C) $x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$

(D) Don't know.

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Applications - vibrations, undamped

- Undamped mass spring

$$mx'' + kx = 0$$

$$mr^2 + k = 0$$

$$r = \pm\sqrt{\frac{k}{m}}i$$

$$x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

- Natural frequency
  - increases with stiffness
  - decreases with mass

Applications - vibrations, undamped

Trig identity reminders

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

$$2\cos(3t+\pi/3) =$$

(A) $2\sin(\pi/3)\cos(3t) - 2\sin(\pi/3)\cos(3t)$

(B) $2\sin(\pi/3)\cos(3t) + 2\sin(\pi/3)\cos(3t)$

☆ (C) $2\cos(\pi/3)\cos(3t) - 2\sin(\pi/3)\sin(3t)$

$= \cos(3t) - \sqrt{3}\sin(3t)$

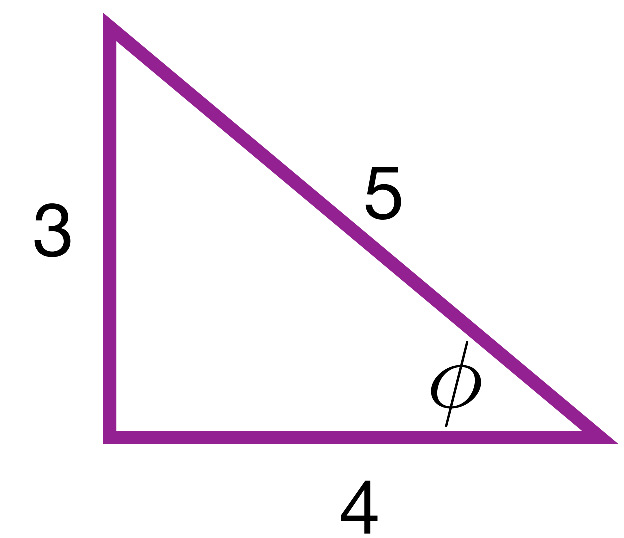
(D) $2\cos(\pi/3)\cos(3t) + 2\sin(\pi/3)\sin(3t)$

(E) Don't know / still thinking.

Applications - vibrations, undamped

- Converting from sum-of-sin-cos to a single cos expression:
  - Example:

$$
\begin{aligned}
&4\cos(2t) + 3\sin(2t) \\
&= 5\left(\frac{4}{5}\cos(2t) + \frac{3}{5}\sin(2t)\right) \\
&= 5(\cos(\phi)\cos(2t) + \sin(\phi)\sin(2t)) \\
&= 5\cos(2t - \phi)
\end{aligned}
$$



$$\phi = 0.9273$$

$$4^2 + 3^2 = 5^2$$

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

(cos(A), sin(A)) must lie on the unit circle. i.e. $\cos^2(A)+\sin^2(A) = 1$.

Applications - vibrations, undamped

- Converting from sum-of-sin-cos to a single cos expression:

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

  - Step 1 - Factor out $A = \sqrt{C_1^2 + C_2^2}$.
  - Step 2 - Find the angle ϕ for which $\cos(\phi) = \dfrac{C_1}{\sqrt{C_1^2 + C_2^2}}$ and $\sin(\phi) = \dfrac{C_2}{\sqrt{C_1^2 + C_2^2}}$.
  - Step 3 - Rewrite the solution as $y(t) = A\cos(\omega_0 t - \phi)$.

Applications - vibrations, damped

- Damped mass-spring

$$mx'' + \gamma x' + kx = 0 \qquad\qquad m, \gamma, k > 0$$

$$\Rightarrow \quad mr^2 + \gamma r + k = 0$$

$$r_{1,2} = -\frac{\gamma}{2m} \pm \frac{\sqrt{\gamma^2 - 4km}}{2m} \quad = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

negative or complex

smaller than 1 or complex

(A) Always complex roots.

(B) Always real roots.

(C) Always one +, one - root.

☆ (D) Never exp growth.

(E) Don't know / still thinking.

There are three cases...

Applications - vibrations, damped

- Damped oscillations

$$r_{1,2} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

(i) $\frac{4km}{\gamma^2} < 1 \quad \Rightarrow$ r_1, r_2 < 0, exponential decay only

(over damped - γ large)

(ii) $\frac{4km}{\gamma^2} = 1 \quad \Rightarrow$ r_1=r_2, exp and t*exp decay

(critically damped)

(iii) $\frac{4km}{\gamma^2} > 1 \quad \Rightarrow \quad r = \alpha \pm \beta i$

$$\alpha = -\frac{\gamma}{2m} < 0 \Rightarrow$$ decaying oscillations

(under damped - γ small)

$$x(t) = e^{\alpha t}\left(C_1 \cos(\beta t) + C_2 \sin(\beta t)\right)$$

$$\beta = \sqrt{\frac{4km}{\gamma^2} - 1}$$

← called pseudo-frequency

For graphs, see:
https://www.desmos.com/calculator/8v1nueimow

Forced vibrations

- Newton's 2nd Law:

$$ma = -kx - \gamma v + F(t)$$

spring force — drag force — applied/external force

$$mx'' + \gamma x' + kx = F(t)$$

- Forced vibrations - nonhomogeneous linear equation with constant coefficients.

- Building during earthquake, tuning fork near instrument, car over washboard road, polar bond under EM stimulus (classical, not quantum).

Forced vibrations, no damping

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

- For what value(s) of *w* does this equation have an unbounded solution?

★ (A) w = sqrt(k/m)

(B) w = m/F_0

(C) w = $(k/m)^2$

(D) w = 2π

Forced vibrations, no damping, away from w_0

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

$$x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \qquad \omega_0 = ?\sqrt{\frac{k}{m}}$$

natural frequency

- Case 1: $\omega \neq \omega_0$

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$A = ?, B = ?$$

Forced vibrations, no damping, away from w_0

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

$$x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

natural frequency

- Case 1: $\omega \neq \omega_0$

$$x_p(t) = A\cos(\omega t) + B\sin(\omega t)$$

$$x_p''(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

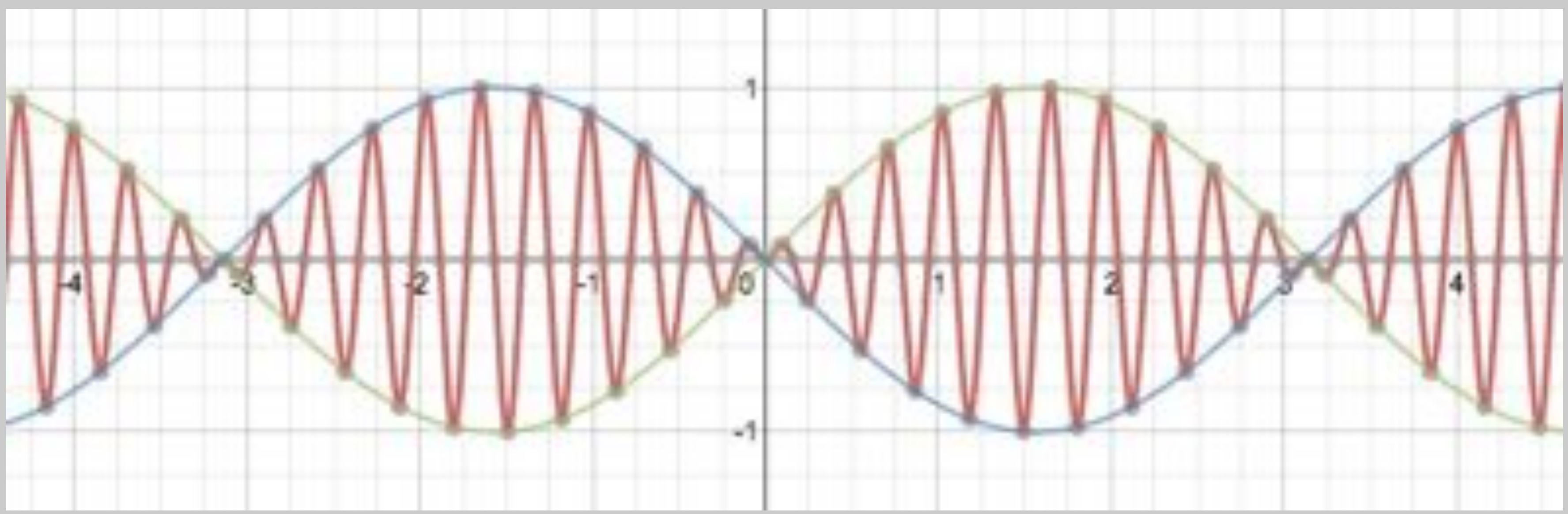
$$mx_p'' + kx_p = (k - \omega^2 m)A\cos(\omega t) + (k - \omega^2 m)B\sin(\omega t)$$

$$= F_0\cos(\omega t) \Rightarrow A = \frac{F_0}{(k - \omega^2 m)}, B = 0$$

Forced vibrations, no damping, away from w$_0$

- Without damping ($\gamma = 0$), $\omega \neq \omega_0$.
  - Simp ... $C_2 = 0.$

When w$_0$≈w --> beats



$$x(t) = \underbrace{\frac{2F_0}{m(\omega_0^2 - \omega^2)} \sin\left(\frac{(\omega_0 - \omega)t}{2}\right)}_{\text{amplitude envelope}} \sin\left(\frac{(\omega_0 + \omega)t}{2}\right)$$

amplitude envelope

https://www.desmos.com/calculator/cfjfpxef1w

Forced vibrations, no damping, $w=w_0$

- Without damping ($\gamma = 0$), $\omega = \omega_0$.

$$mx'' + kx = F_0 \cos(\omega t)$$

$$x'' + \omega_0^2 x = \frac{F_0}{m} \cos(\omega_0 t)$$

$$\omega_0 = \sqrt{\frac{k}{m}}$$

$$x_p(t) = t(A\cos(\omega_0 t) + B\sin(\omega_0 t))$$

$$x_p'(t) = A\cos(\omega_0 t) + B\sin(\omega_0 t)$$

$$+t(-\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t))$$

RHS solves the homogenous equation:

$$r^2 + \omega_0^2 = 0$$

$$r = \pm\omega_0 i$$

$$x_p''(t) = -\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t)$$

$$+(-\omega_0 A\sin(\omega_0 t) + \omega_0 B\cos(\omega_0 t))$$

$$+t(-\omega_0^2 A\cos(\omega_0 t) - \omega_0^2 B\sin(\omega_0 t))$$

$$A = 0$$

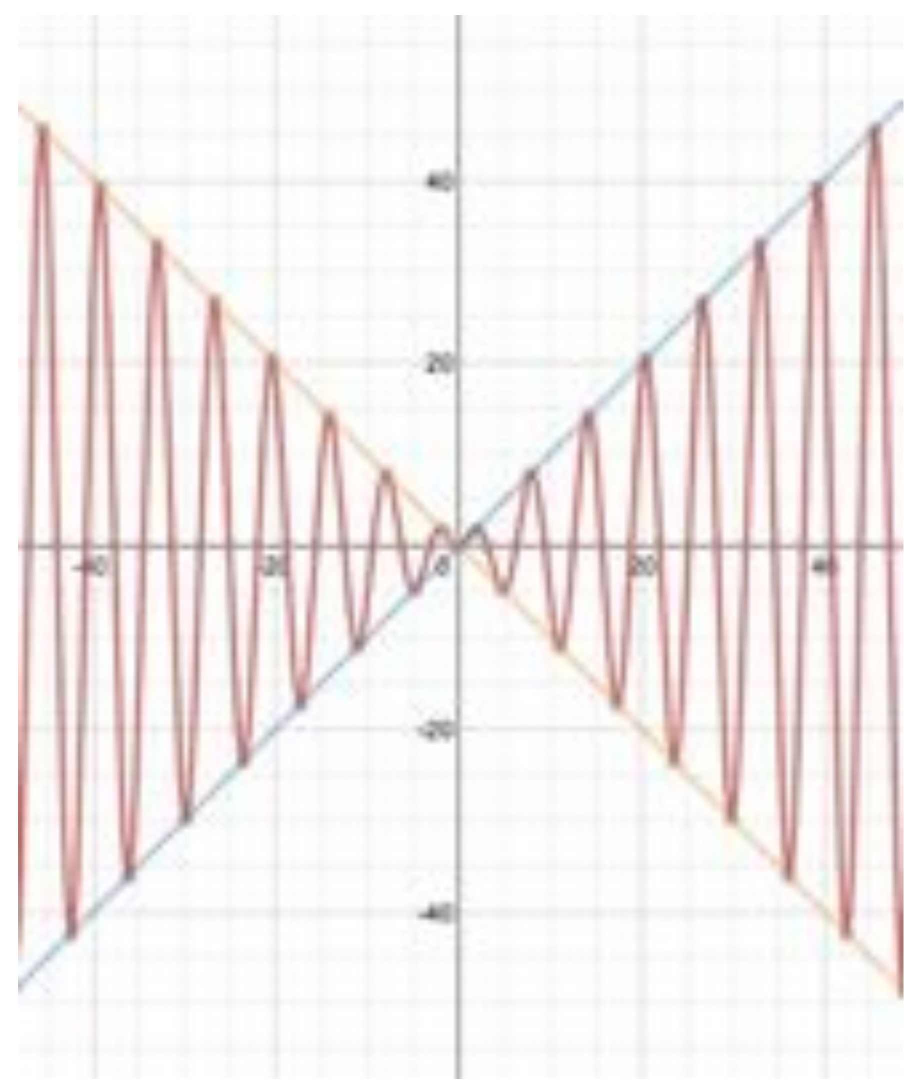
$$B = \frac{F_0}{2\omega_0 m} = \frac{F_0}{2\sqrt{km}}$$

$$x_p(t) = \frac{F_0}{2\sqrt{km}} t\sin(\omega_0 t)$$

Forced vibrations, no damping, $w=w_0$

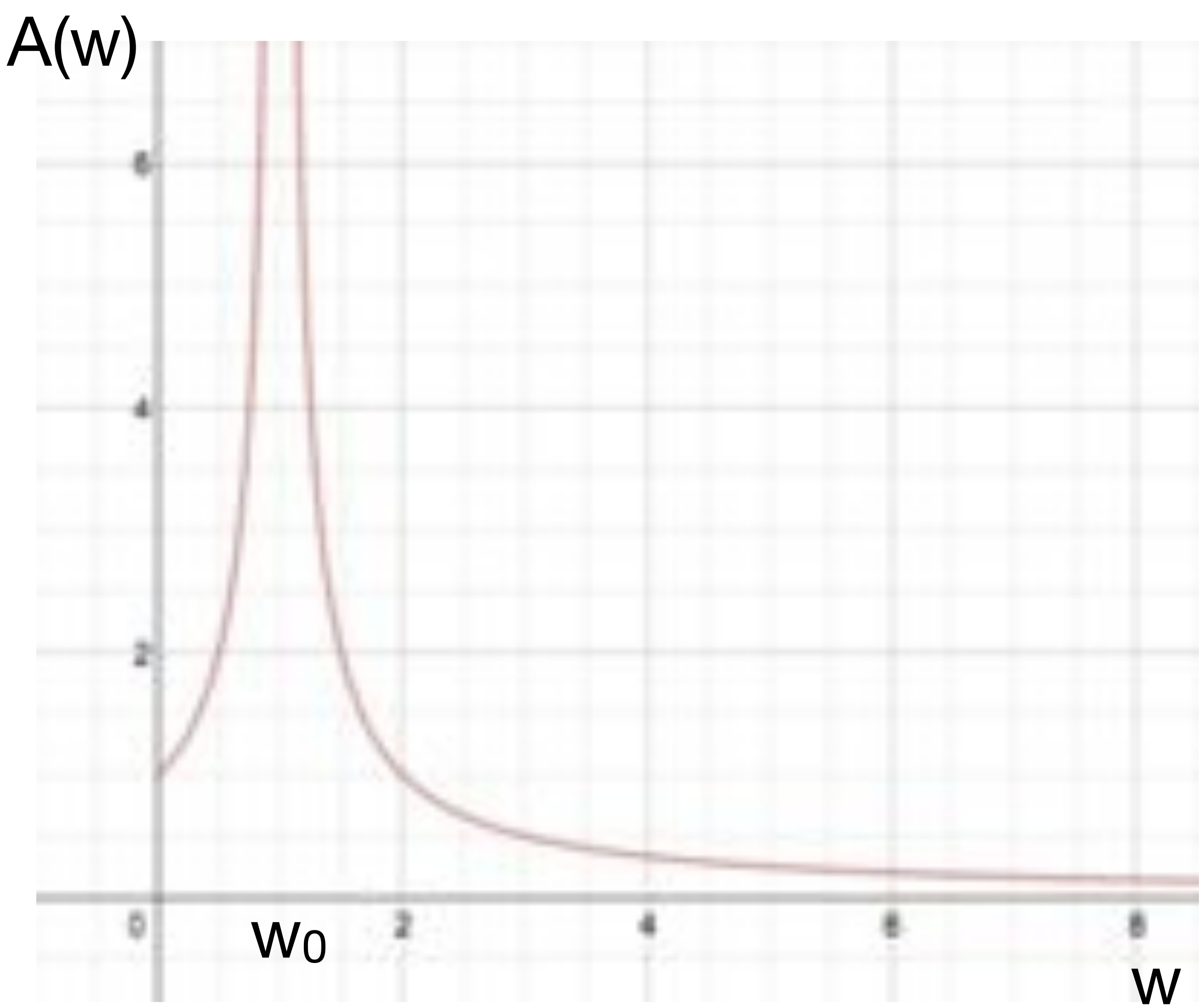
- Without damping ($\gamma = 0$), $\omega = \omega_0$.
  - Long term behaviour - x_p grows unbounded, swamping out x_h.

$$x_p(t) = \frac{F_0}{2\sqrt{km}} t \sin(\omega_0 t)$$



Forced vibrations, no damping, summary

- Plot of the amplitude of the particular solution as a function of ω.



- Calculated:

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

- Plotted with:

$$\frac{F_0}{m} = 1, \;\; w_0 = 1$$

$$A(\omega) = \frac{1}{|\omega_0^2 - \omega^2|}$$

- Recall that for $\omega = \omega_0$, the amplitude grows without bound.

Forced vibrations, with damping

$$m x'' + \gamma x' + kx = F_0 \cos \omega t$$

$$x'' + c x' + \omega_0^2 x = \frac{F_0}{m} \cos \omega t$$

No conflict with $x_h(t)$!

$$x_p = A \cos \omega t + B \sin \omega t$$

$$x_p' = -\omega A \sin \omega t + \omega B \cos \omega t$$

$$x_p'' = -\omega^2 A \cos \omega t - \omega^2 B \sin \omega t$$

$$-\omega^2 A \cos \omega t - \omega^2 B \sin \omega t + c(-\omega A \sin \omega t + \omega B \cos \omega t) + \omega_0^2 (A \cos \omega t + B \sin \omega t) = \frac{F_0}{m} \cos \omega t$$

$$\underbrace{(-\omega^2 A + c\omega B + \omega_0^2 A)}_{F_0/m} \cos \omega t + \underbrace{(-\omega^2 B - c\omega A + \omega_0^2 B)}_{0} \sin \omega t = \frac{F_0}{m} \cos \omega t$$

$$A = \frac{F_0}{m} \frac{\omega_0^2 - \omega^2}{(c\omega)^2 + (\omega_0^2 - \omega^2)}$$

$$B = \frac{F_0}{m} \frac{c\omega}{(c\omega)^2 + (\omega_0^2 - \omega^2)}$$

$$x_p(t) = \frac{F_0}{m} \cdot \frac{1}{\sqrt{(c\omega)^2 + (\omega_0^2 - \omega^2)^2}} \left(\frac{(\omega_0^2 - \omega^2)}{\sqrt{(c\omega)^2 + (\omega_0^2 - \omega^2)^2}} \cos \omega t + \frac{c\omega}{\sqrt{(c\omega)^2 + (\omega_0^2 - \omega^2)^2}} \sin \omega t \right)$$

Forced vibrations, with damping