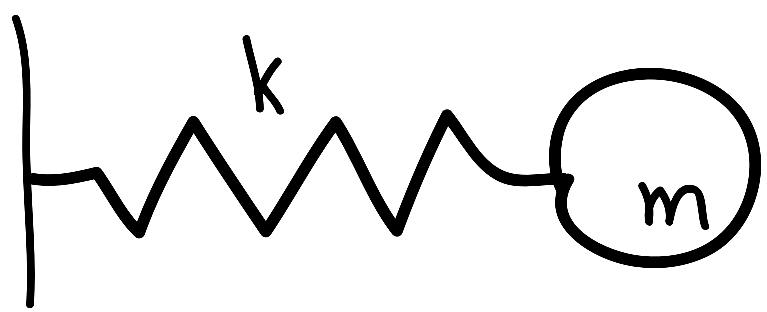
Today

- Mass-springs as models for everything.
- Forced vibrations
    - Newton’s 2nd Law with external forcing.
    - Forced mass-spring system without damping away from resonance.
    - Forced mass-spring system without damping at resonance.
    - Forced mass-spring system with damping.
- Midterm (Feb 2, in class) - Everything up to and including Tuesday Jan 26 (Method of Undetermined Coefficients).

Applications - vibrations

Mass-spring systems



Applications - vibrations

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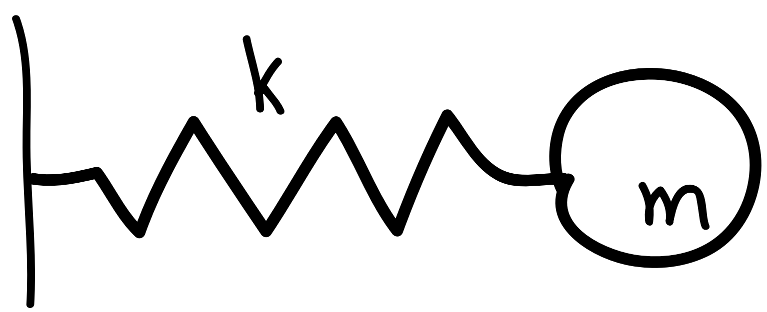

$ma = F$

Applications - vibrations

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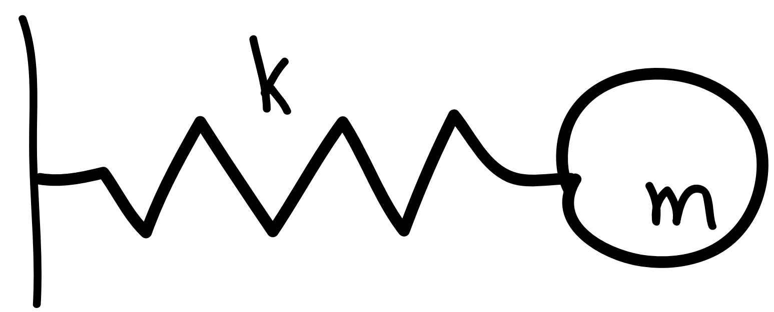

$$ma = F$$

$$E = \frac{1}{2}k(x - x_0)^2$$

Applications - vibrations

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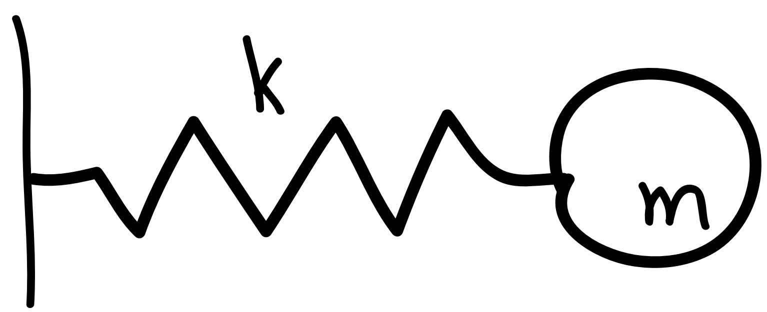

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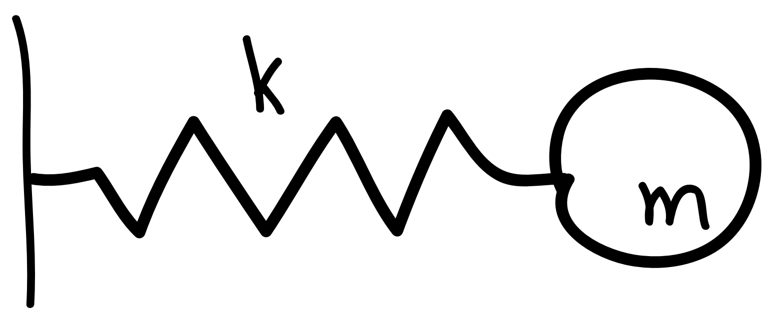

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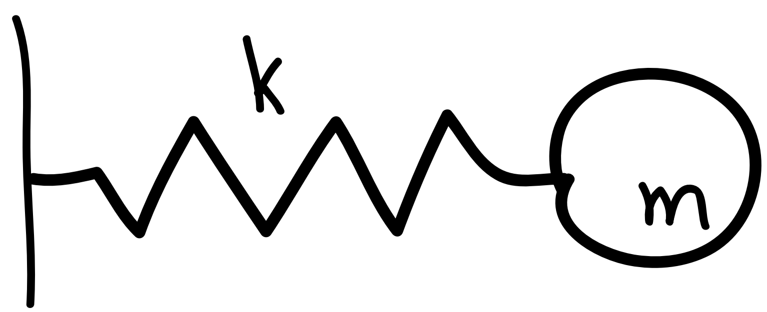

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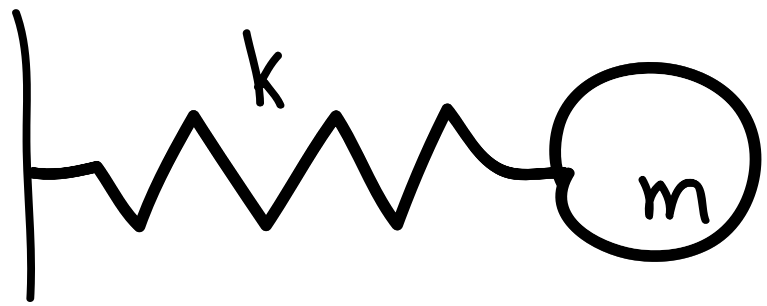

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$$mx'' + kx = kx_0$$

Applications - vibrations

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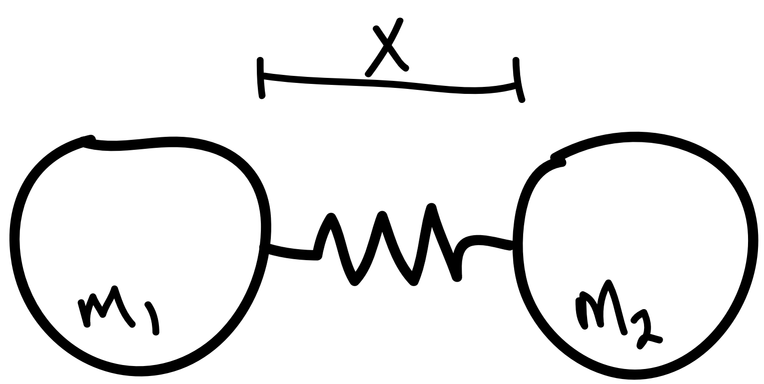
Molecular bonds

Applications - vibrations

Molecular bonds



Applications - vibrations

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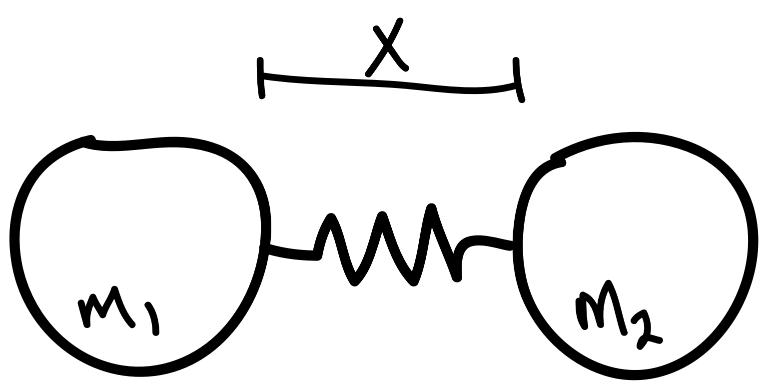
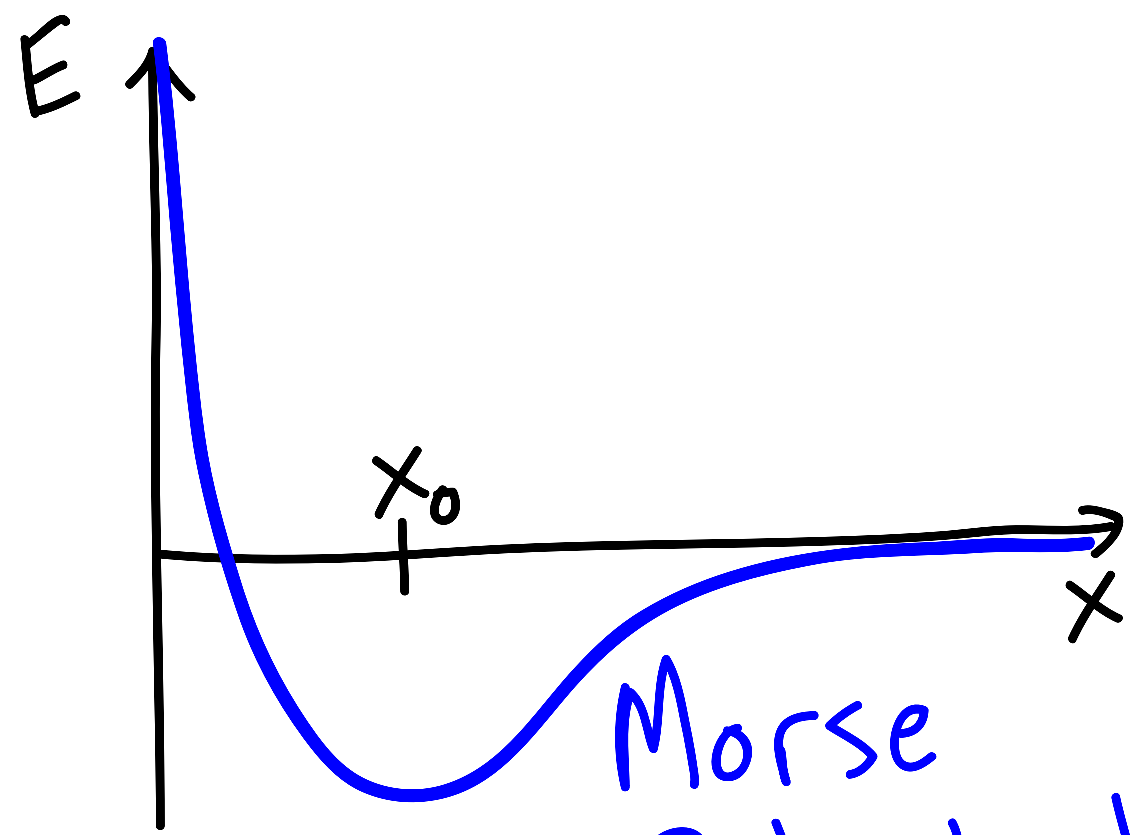
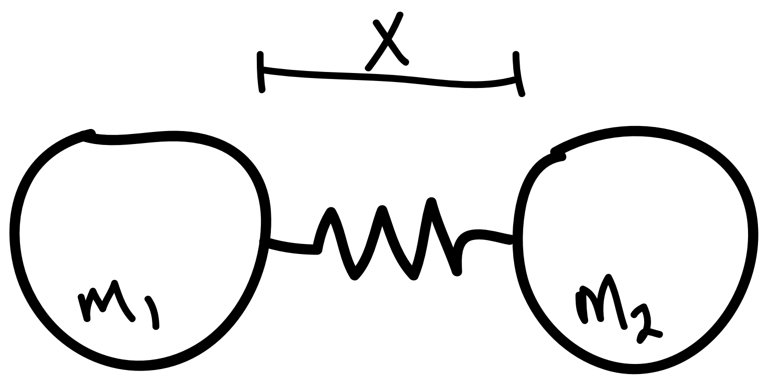


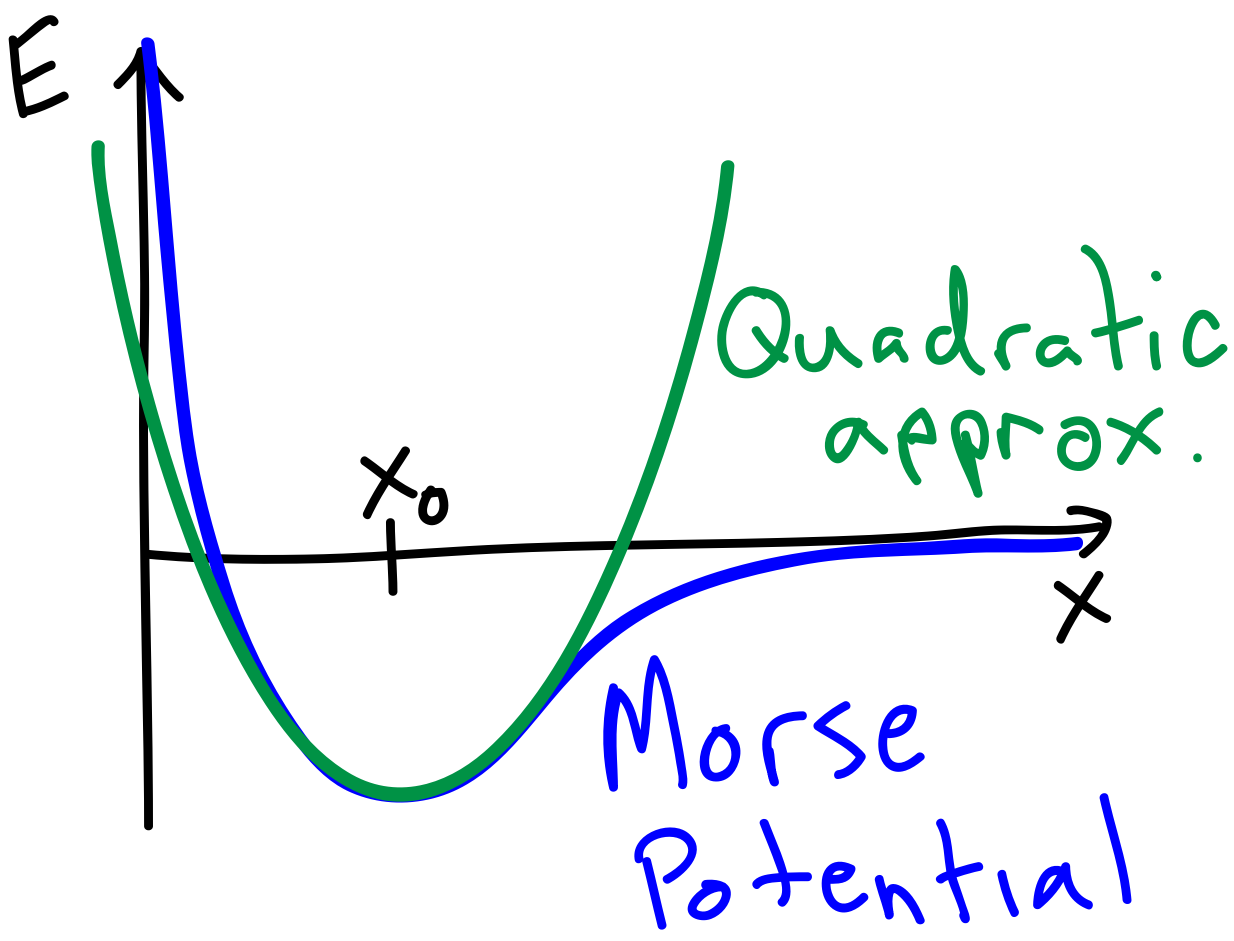


Morse Potential

Applications - vibrations

Molecular bonds

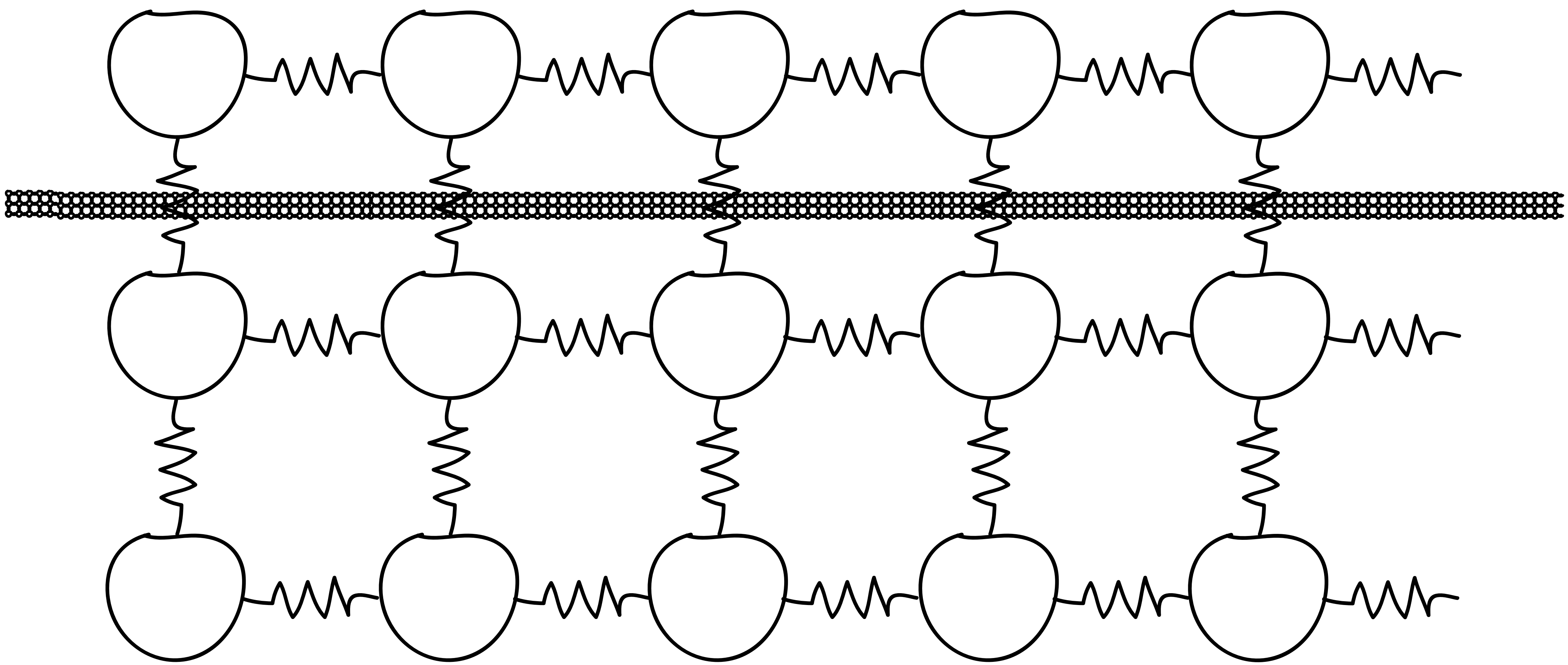




Applications - vibrations

Solid mechanics

e.g. tuning fork, bridges, buildings



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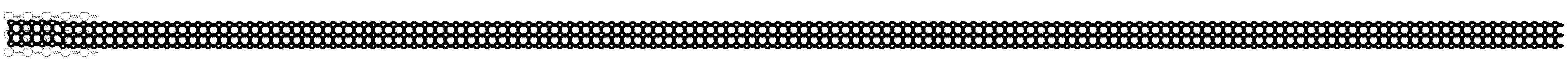

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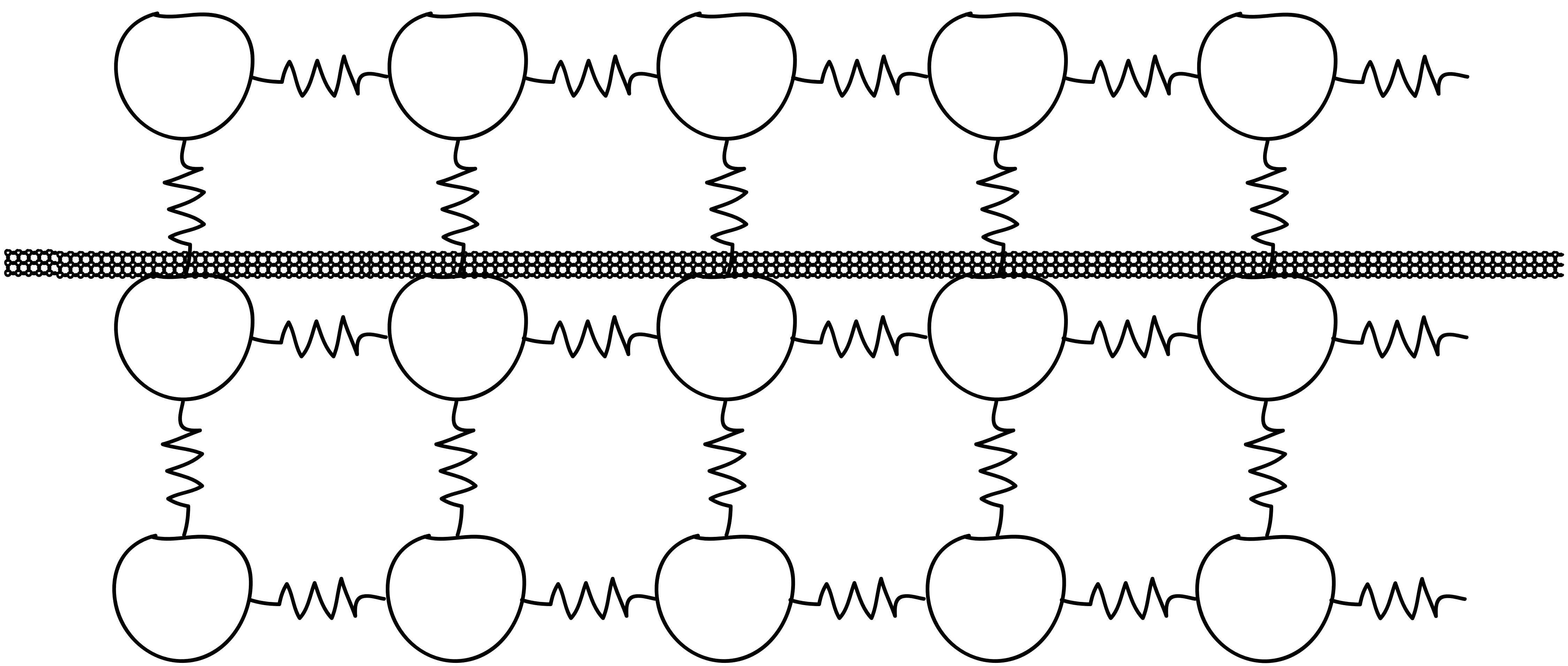

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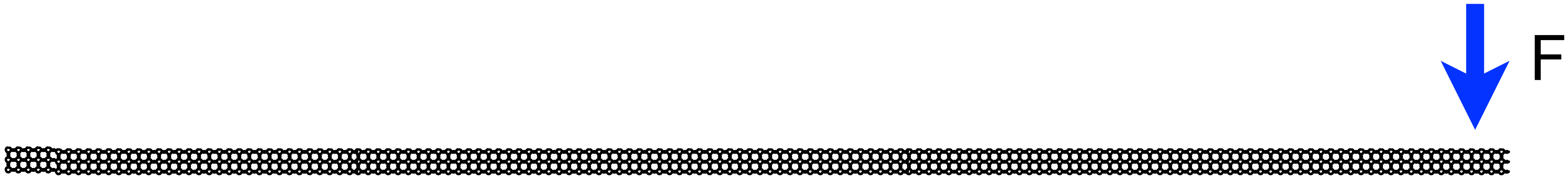

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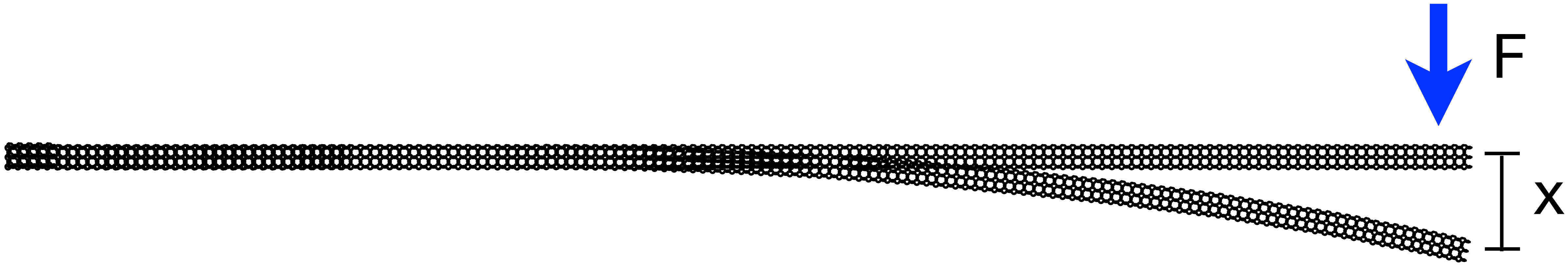

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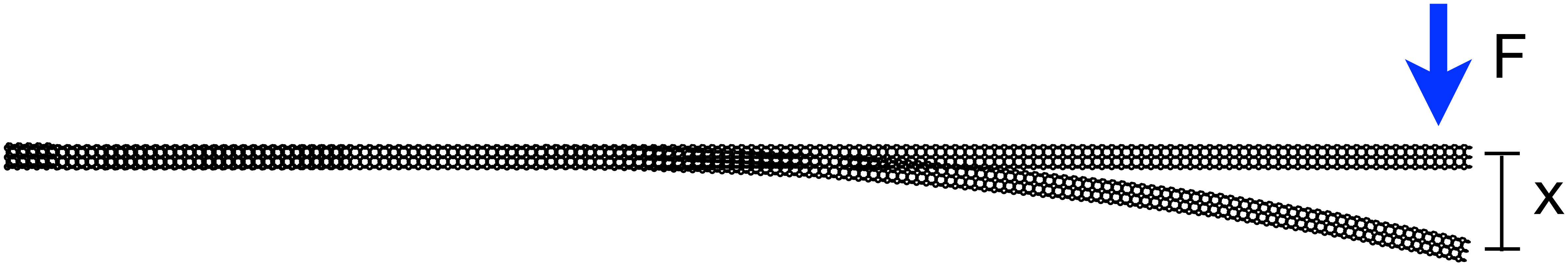

$$x'' = -Kx$$

where K depends on the molecular details of the material and the cross-sectional geometry of the rod.

Applications - vibrations

- So far, no x’ term so no exponential decay in the solutions.

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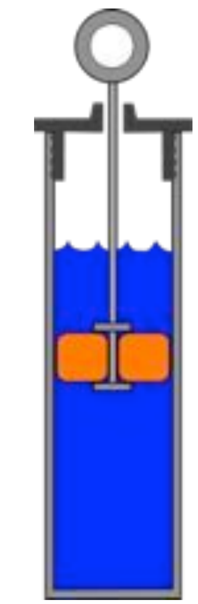
  - sometimes an abstraction that accounts for energy loss.



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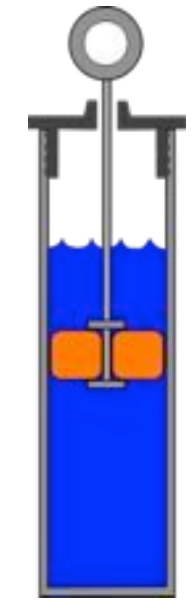

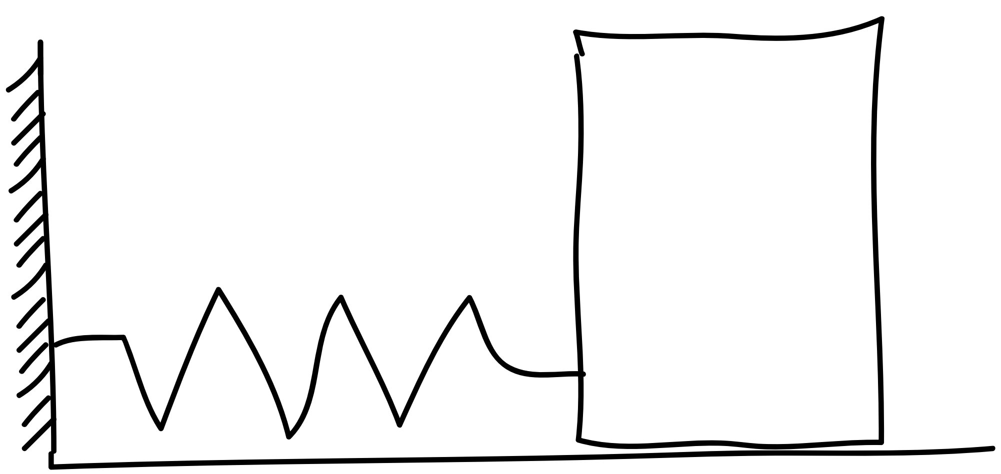

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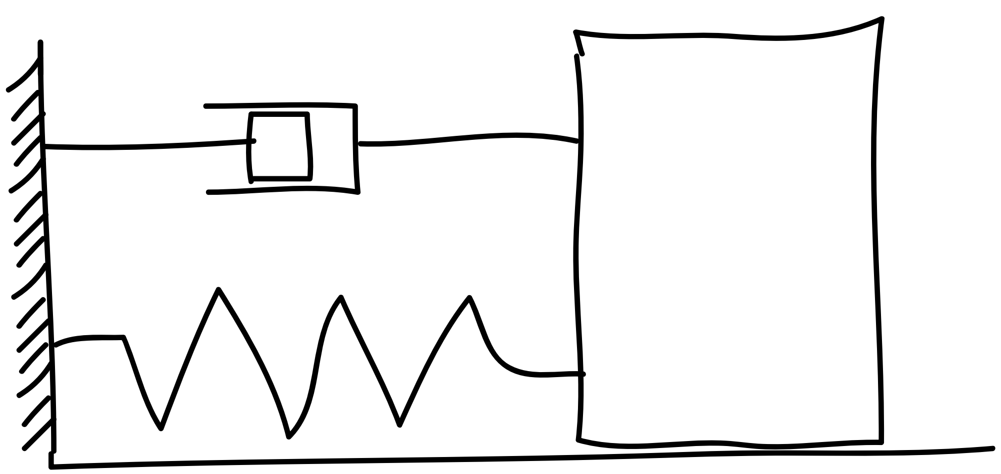

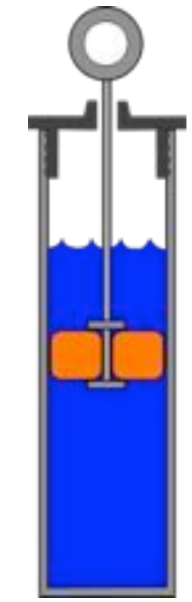

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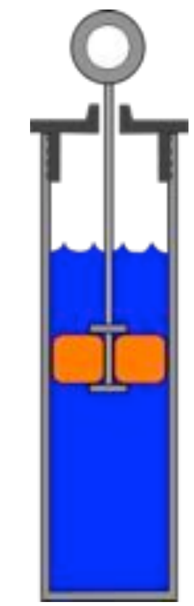
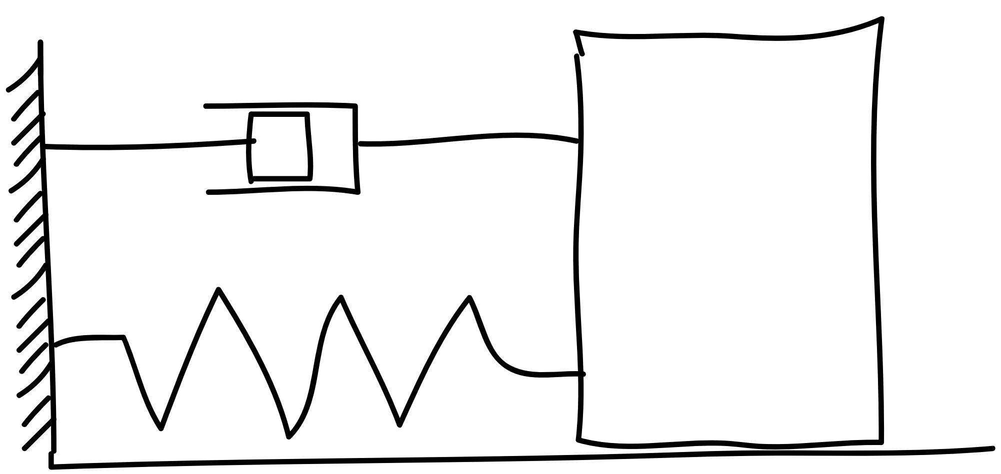
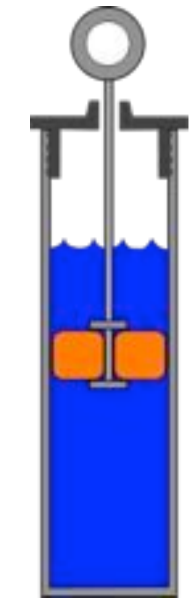




Kelvin-Voigt model

Applications - vibrations



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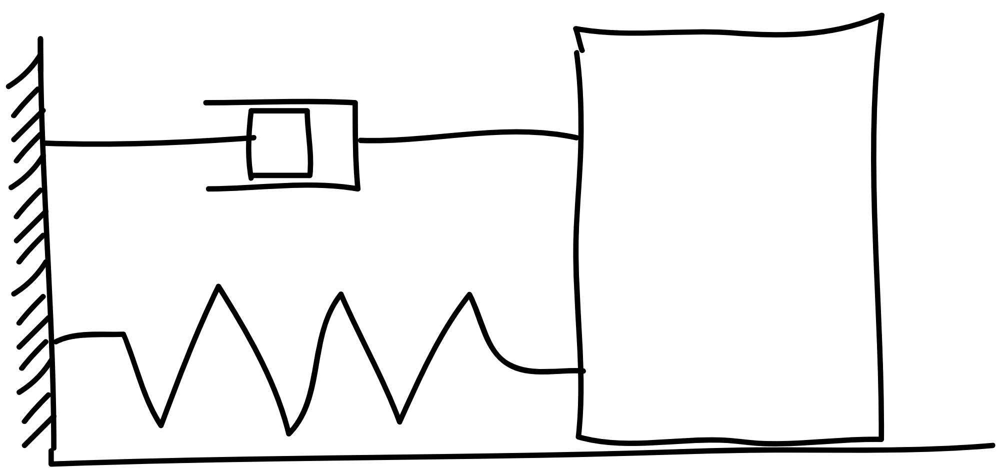

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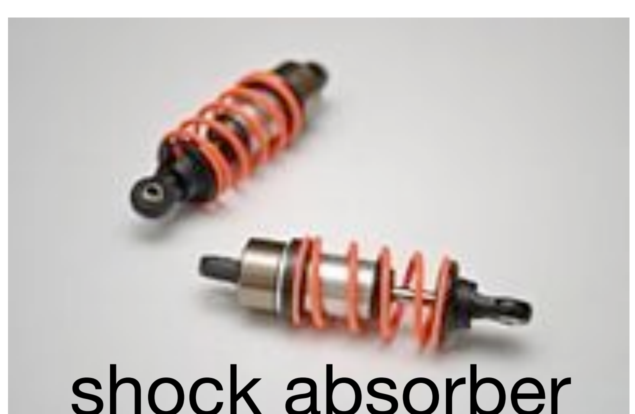

shock absorber

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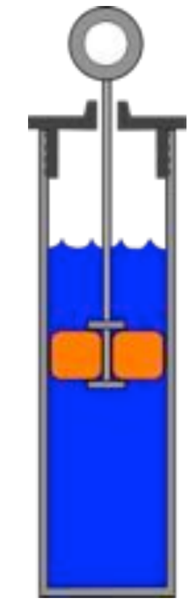
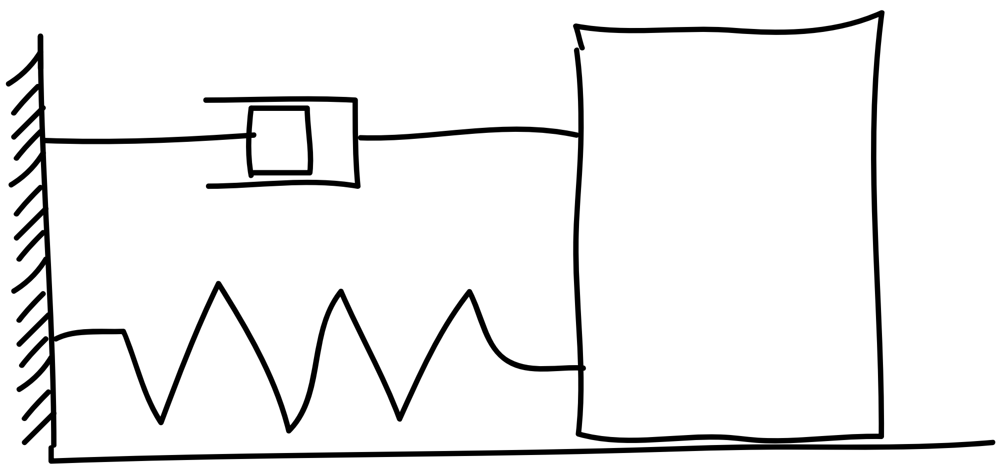
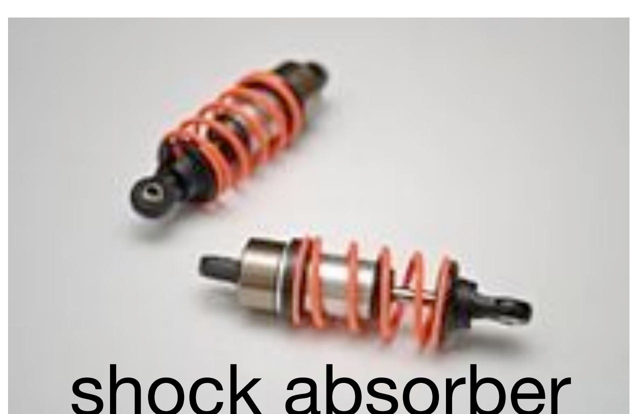




Kelvin-Voigt model

$$ma = -k(x - x_0) - \gamma v$$



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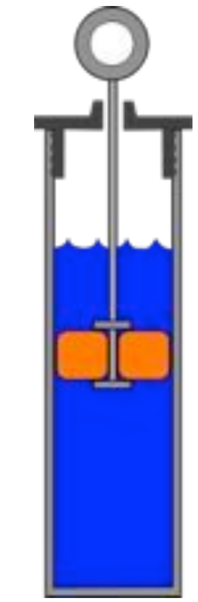

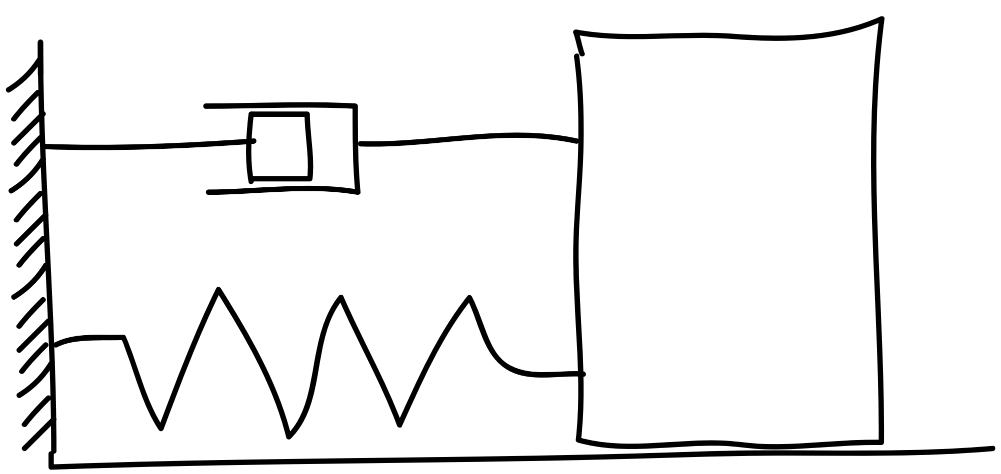

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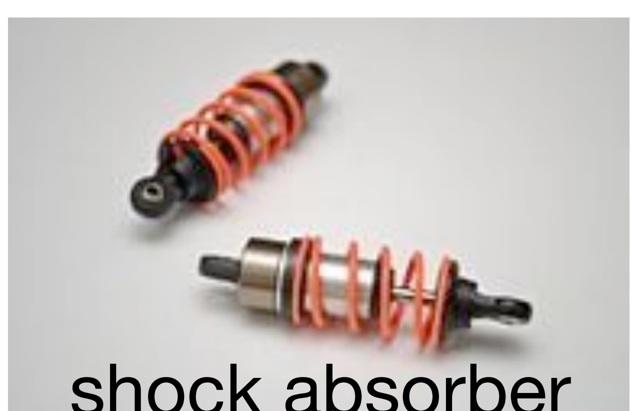

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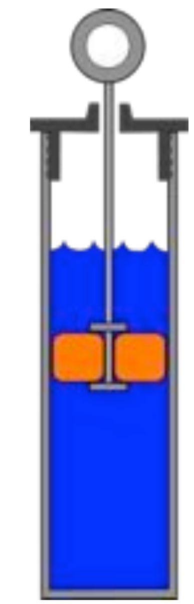

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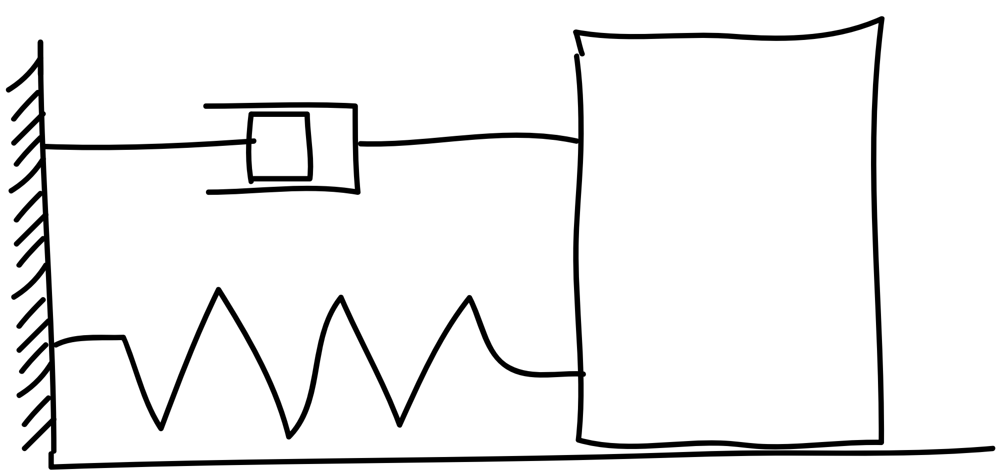

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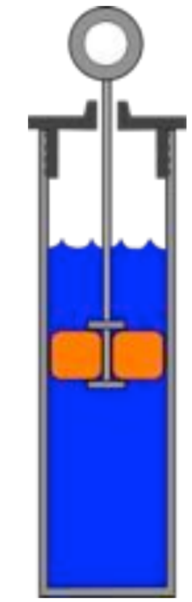

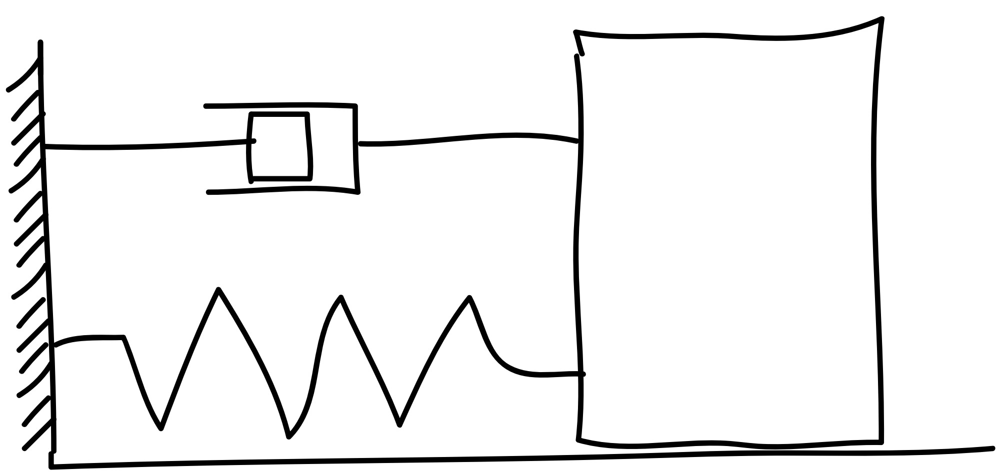

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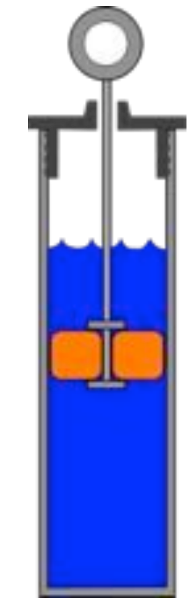

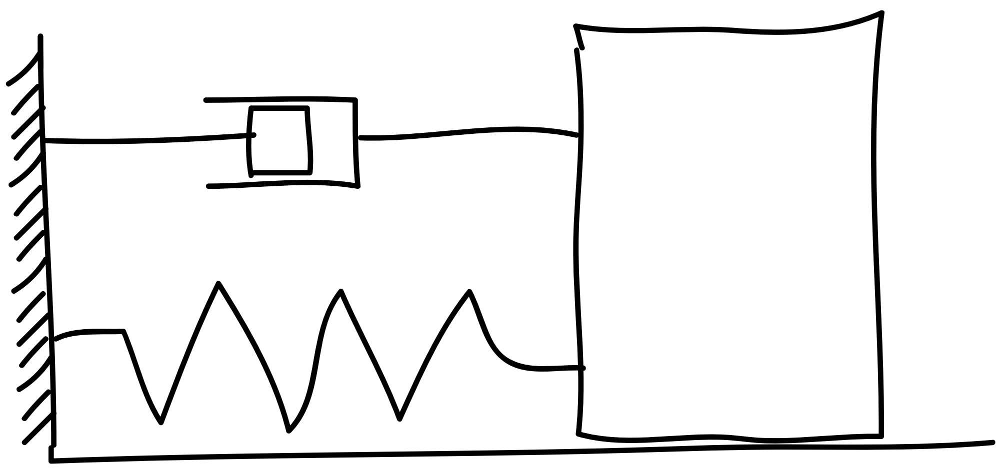

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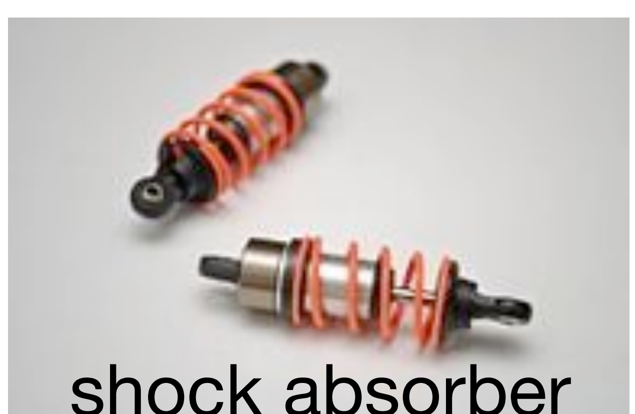
$$my'' + \gamma y' + ky = 0$$



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Applications - forced vibrations

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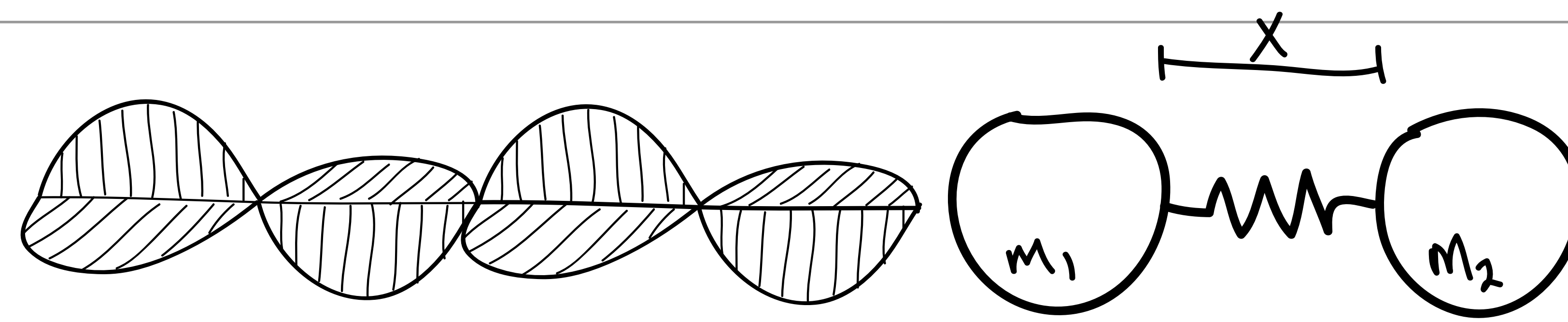
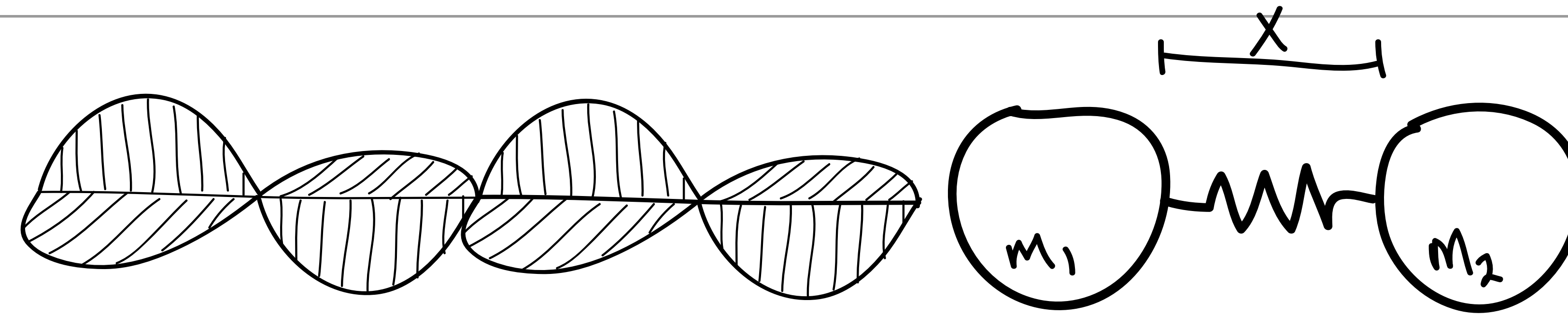


- light hitting a molecular bond

Applications - forced vibrations



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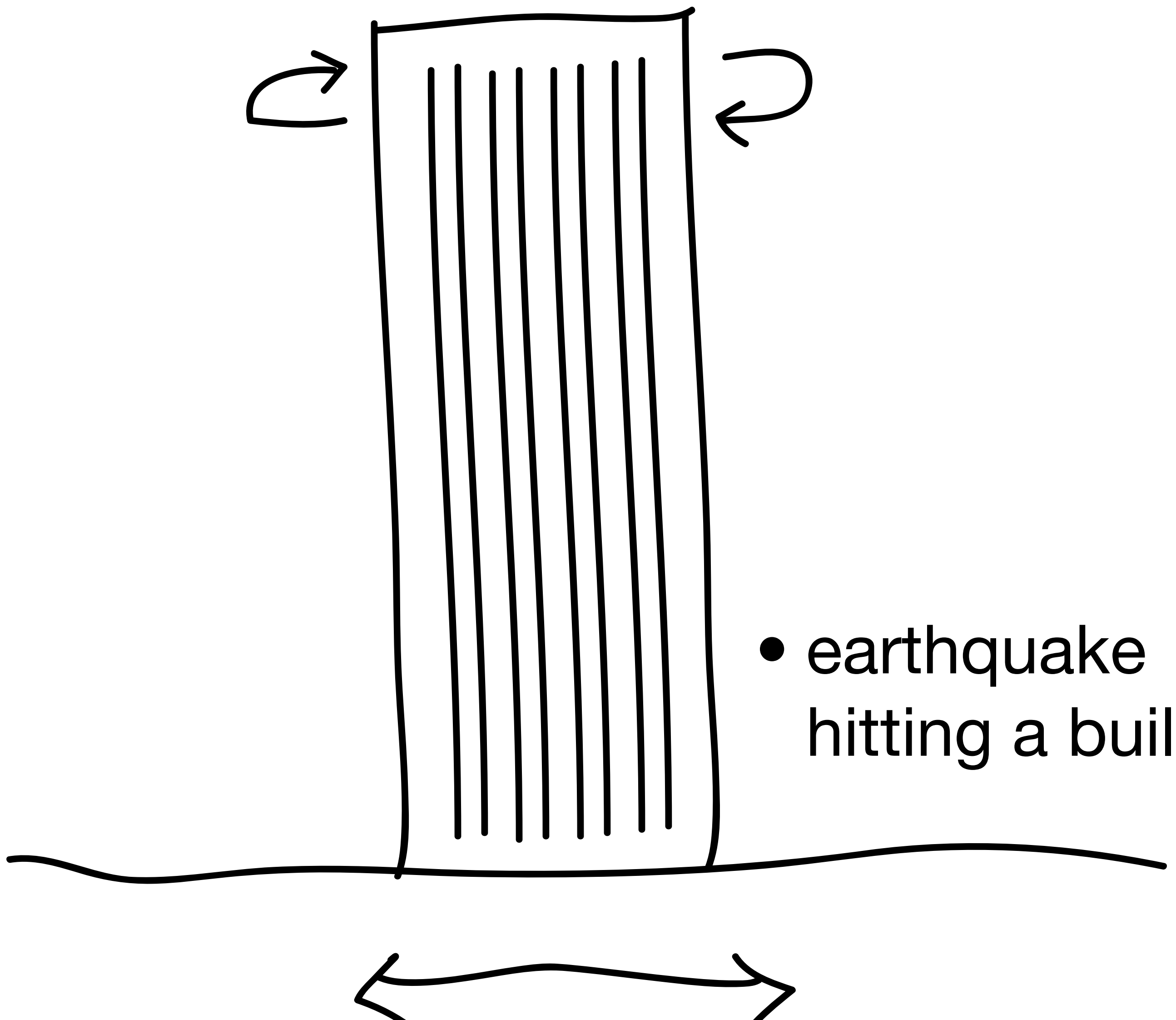
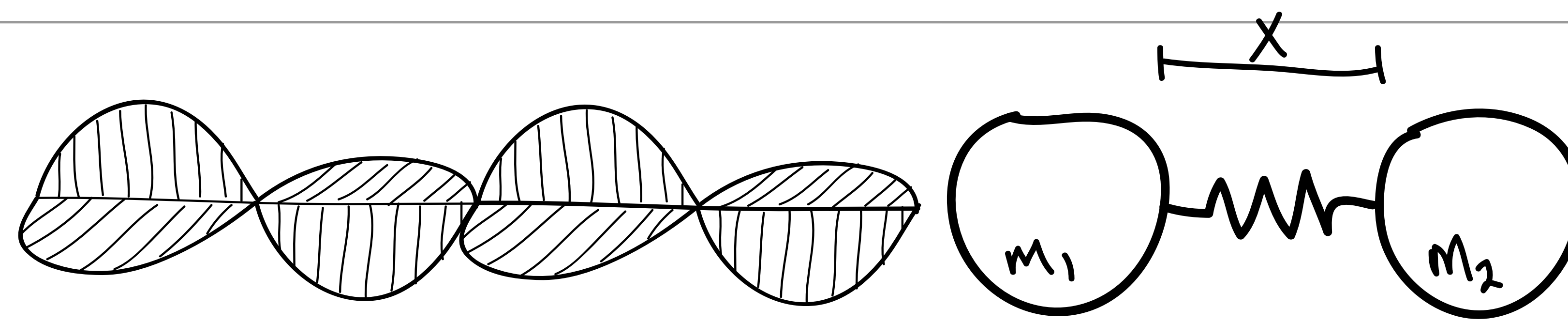


- earthquake
  hitting a building

Applications - forced vibrations



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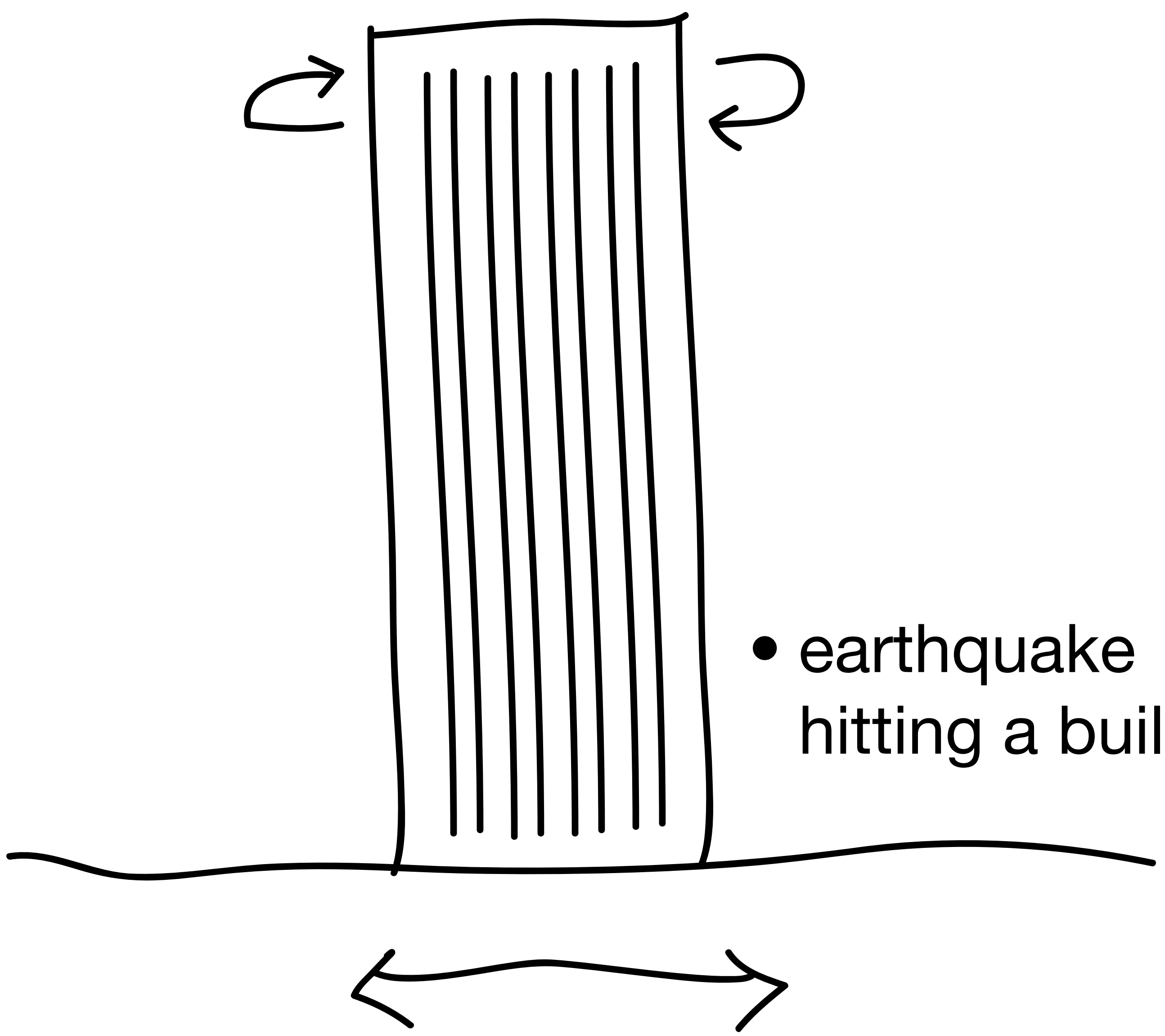

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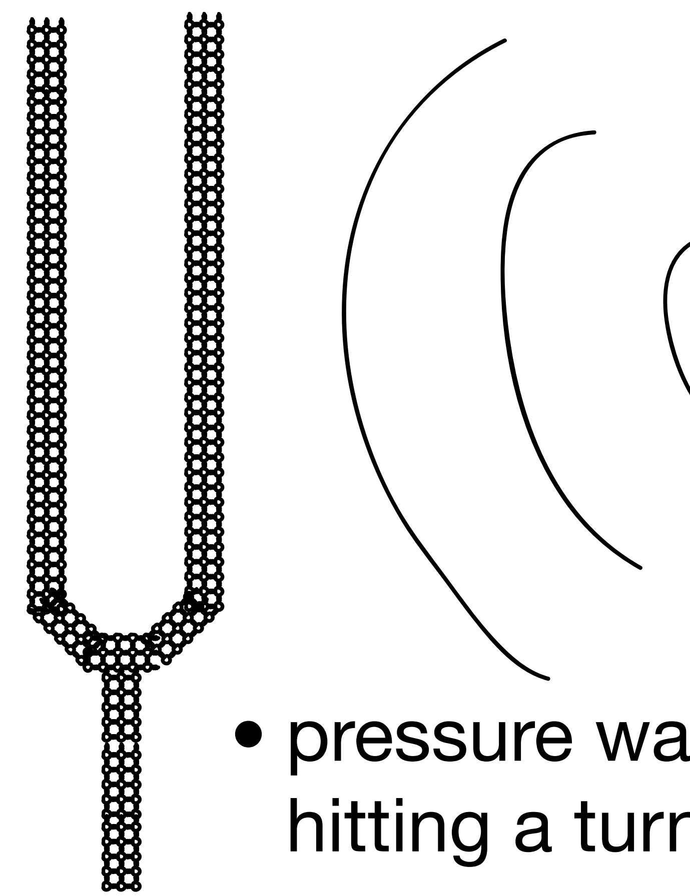

- pressure waves (sound) hitting a turning fork.

Applications - vibrations, undamped

- Undamped mass spring

$$mx'' + kx = 0$$

Applications - vibrations, undamped

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(A) $x(t) = C_1 e^{-\omega_0 t} + C_2 e^{\omega_0 t}$

(B) $x(t) = C_1 e^{-\omega_0 t} + C_2 t e^{-\omega_0 t}$

(C) $x(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$

(D) Don’t know.

$$\omega_0 = \sqrt{\frac{k}{m}}$$

Applications - vibrations, undamped

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- Natural frequency
    - increases with stiffness
    - decreases with mass

Applications - vibrations, undamped

Trig identity reminders

$$\sin(A+B) = \sin(A)\cos(B) + \cos(A)\sin(B)$$
$$\cos(A+B) = \cos(A)\cos(B) - \sin(A)\sin(B)$$

Applications - vibrations, undamped

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$$2\cos(3t + \pi/3) =$$

(A) $2\sin(\pi/3)\cos(3t) - 2\sin(\pi/3)\cos(3t)$

(B) $2\sin(\pi/3)\cos(3t) + 2\sin(\pi/3)\cos(3t)$

(C) $2\cos(\pi/3)\cos(3t) - 2\sin(\pi/3)\sin(3t)$

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(E) Don't know / still thinking.

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$$2\cos(3t+\pi/3) =$$

☆ $$2\cos(\pi/3)\cos(3t) - 2\sin(\pi/3)\sin(3t)$$
$$= \cos(3t) - \sqrt{3}\sin(3t)$$

Applications - vibrations, undamped

- Converting from sum-of-sin-cos to a single cos expression:
  - Example:

$$4\cos(2t) + 3\sin(2t)$$

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

Applications - vibrations, undamped

- Converting from sum-of-sin-cos to a single cos expression:
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Applications - vibrations, undamped

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$$\cos(A-B) = \overset{\cancel{4}}{\cos(A)}\cos(B) + \overset{\cancel{3}}{\sin(A)}\sin(B)$$

(cos(A), sin(A)) must lie on the unit circle. i.e. $\cos^2(A)+\sin^2(A) = 1$.

Applications - vibrations, undamped

- Converting from sum-of-sin-cos to a single cos expression:
  - Example:

$$4\cos(2t) + 3\sin(2t)$$

$$4^2 + 3^2 = 5^2$$

$$\cancel{4} \qquad\qquad \cancel{3}$$

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

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- Converting from sum-of-sin-cos to a single cos expression:
  - Example:

$$4\cos(2t) + 3\sin(2t)$$
$$= 5\left(\frac{4}{5}\cos(2t) + \frac{3}{5}\sin(2t)\right)$$

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$$\qquad\;\; \cancel{4} \qquad\qquad\qquad\qquad\quad \cancel{3}$$
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  - Example:

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Applications - vibrations, undamped

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$$
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&\quad = 5\cos(2t - \phi)
\end{aligned}
$$

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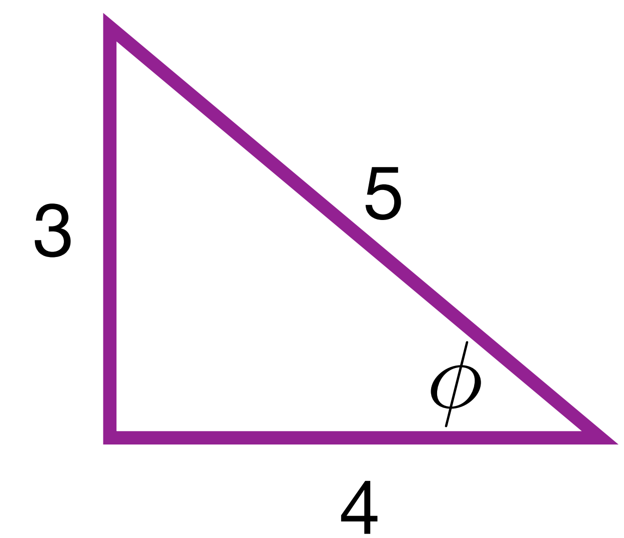

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Applications - vibrations, undamped

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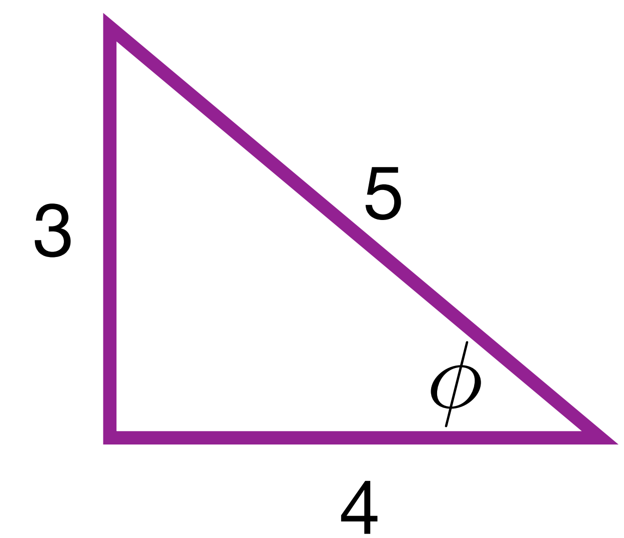

$$\phi = 0.9273$$

$$\cos(A - B) = \cos(A)\cos(B) + \sin(A)\sin(B)$$

Applications - vibrations, undamped

- Converting from sum-of-sin-cos to a single cos expression:

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

Applications - vibrations, undamped

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  - Step 1 - Factor out $A = \sqrt{C_1^2 + C_2^2}$.

Applications - vibrations, undamped

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$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

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  - Step 2 - Find the angle ϕ for which $\cos(\phi) = \dfrac{C_1}{\sqrt{C_1^2 + C_2^2}}$ and $\sin(\phi) = \dfrac{C_2}{\sqrt{C_1^2 + C_2^2}}$.

Applications - vibrations, undamped

- Converting from sum-of-sin-cos to a single cos expression:

$$y(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

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  - Step 3 - Rewrite the solution as $y(t) = A\cos(\omega_0 t - \phi)$.

Applications - vibrations, damped

- Damped mass-spring

$$mx'' + \gamma x' + kx = 0 \qquad\qquad m, \gamma, k > 0$$

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(A) Always complex roots.

(B) Always real roots.

(C) Always one +, one - root.

(D) Never exp growth.

(E) Don’t know / still thinking.

Applications - vibrations, damped

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$$mx'' + \gamma x' + kx = 0 \qquad\qquad m, \gamma, k > 0$$

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smaller than 1
or complex

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There are three cases...

Applications - vibrations, damped

- Damped oscillations

$$r_{1,2} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

(i) $\frac{4km}{\gamma^2} < 1$

(ii) $\frac{4km}{\gamma^2} = 1$

(iii) $\frac{4km}{\gamma^2} > 1$

Applications - vibrations, damped

- Damped oscillations

$$r_{1,2} = \frac{\gamma}{2m}\left(-1 \pm \sqrt{1 - \frac{4km}{\gamma^2}}\right)$$

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(over damped - γ large)

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$$\alpha = -\frac{\gamma}{2m} < 0$$

Applications - vibrations, damped

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Applications - vibrations, damped

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Applications - vibrations, damped

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$\beta = \sqrt{\frac{4km}{\gamma^2} - 1}$ ⟵ called pseudo-frequency

Applications - vibrations, damped

- Damped oscillations

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For graphs, see:
https://www.desmos.com/calculator/8v1nueimow

Forced vibrations

- Newton's 2nd Law:

$$ma = -kx - \gamma v + F(t)$$

Forced vibrations

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$$ma = \underbrace{-kx}_{\text{spring force}} - \gamma v + F(t)$$

Forced vibrations

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Forced vibrations

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$$ma = \underbrace{-kx}_{\text{spring force}} \underbrace{- \gamma v}_{\text{drag force}} \underbrace{+F(t)}_{\text{applied/external force}}$$

Forced vibrations

- Newton's 2nd Law:

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spring force   drag force   applied/external force

$$mx'' + \gamma x' + kx = F(t)$$

Forced vibrations

- Newton's 2nd Law:

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- Forced vibrations - nonhomogeneous linear equation with constant coefficients.

Forced vibrations

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- Forced vibrations - nonhomogeneous linear equation with constant coefficients.

- Building during earthquake, tuning fork near instrument, car over washboard road, polar bond under EM stimulus (classical, not quantum).

Forced vibrations, no damping

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

- For what value(s) of *w* does this equation have an unbounded solution?

(A) w = sqrt(k/m)

(B) w = m/F_0

(C) w = $(k/m)^2$

(D) w = 2π

Forced vibrations, no damping

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Forced vibrations, no damping, away from w_0

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forcing frequency

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Forced vibrations, no damping, away from w0

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$$mx'' + kx = 0$$

Forced vibrations, no damping, away from w0

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forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

$$x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t)$$

Forced vibrations, no damping, away from w_0

- Without damping ($\gamma = 0$).

forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

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$$x_h(t) = C_1 \cos(\omega_0 t) + C_2 \sin(\omega_0 t) \qquad \omega_0 = ?$$

Forced vibrations, no damping, away from w0

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forcing frequency

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Forced vibrations, no damping, away from w$_0$

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forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

$$mx'' + kx = 0$$

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natural frequency

Forced vibrations, no damping, away from w0

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forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

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- Case 1: $\omega \neq \omega_0$

natural frequency

Forced vibrations, no damping, away from w0

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forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

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natural frequency

- Case 1: $\omega \neq \omega_0$

$$x_p(t) = A \cos(\omega t) + B \sin(\omega t)$$

Forced vibrations, no damping, away from w0

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natural frequency

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$$x_p(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$A = ?, B = ?$$

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natural frequency

- Case 1: $\omega \neq \omega_0$

$$x_p(t) = A \cos(\omega t) + B \sin(\omega t)$$

$$x_p''(t) = -\omega^2 A \cos(\omega t) - \omega^2 B \sin(\omega t)$$

Forced vibrations, no damping, away from w0

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forcing frequency

$$mx'' + kx = F_0 \cos(\omega t)$$

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$$x_p''(t) = -\omega^2 A\cos(\omega t) - \omega^2 B\sin(\omega t)$$

$$mx_p'' + kx_p = (k - \omega^2 m)A\cos(\omega t) + (k - \omega^2 m)B\sin(\omega t)$$

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$$= F_0 \cos(\omega t)$$

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$$= F_0 \cos(\omega t) \quad \Rightarrow A = \frac{F_0}{(k - \omega^2 m)}$$

Forced vibrations, no damping, away from w0

- Without damping ($\gamma = 0$). forcing frequency

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$$mx_p'' + kx_p = (k - \omega^2 m)A\cos(\omega t) + (k - \omega^2 m)B\sin(\omega t)$$

$$= F_0\cos(\omega t) \;\; \Rightarrow A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

Forced vibrations, no damping, away from w0

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$$mx_p'' + kx_p = (k - \omega^2 m)A\cos(\omega t) + (k - \omega^2 m)B\sin(\omega t)$$

$$= F_0 \cos(\omega t) \;\Rightarrow A = \frac{F_0}{m(\omega_0^2 - \omega^2)}, B = 0$$

Forced vibrations, no damping, away from w0

- Without damping ($\gamma = 0$), $\omega \neq \omega_0$.
    - Simple case:

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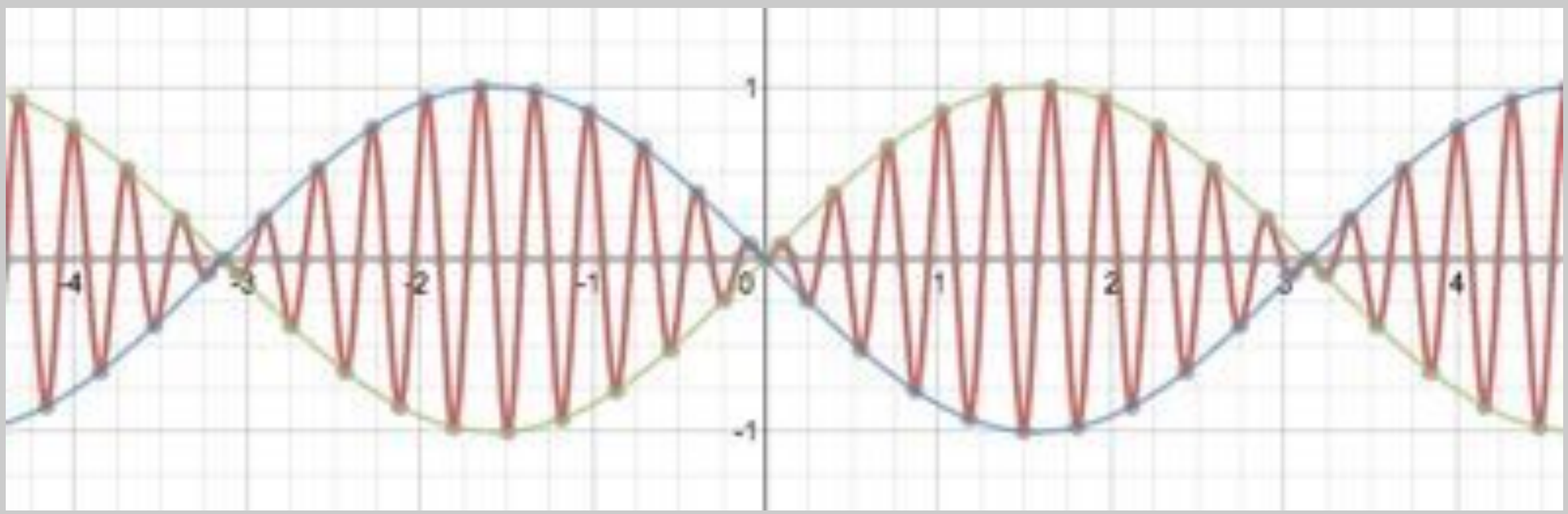
Forced vibrations, no damping, away from w_0

- Without damping ($\gamma = 0$), $\omega \neq \omega_0$.
  - Simp $C_2 = 0.$

When $w_0 \approx w$ --> beats



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Forced vibrations, no damping, away from w$_0$

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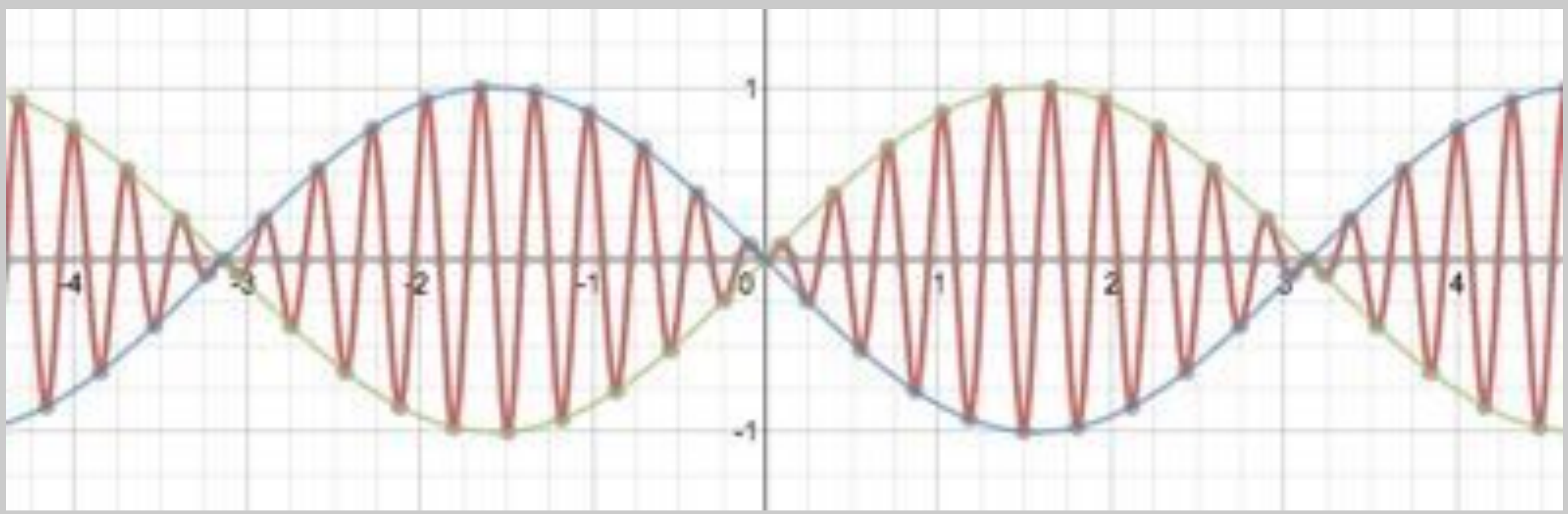
When w$_0$≈w --> beats



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https://www.desmos.com/calculator/cfjfpxef1w

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RHS solves the homogenous equation:

$$r^2 + \omega_0^2 = 0$$

$$r = \pm\omega_0 i$$

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$$A = 0$$

Forced vibrations, no damping, $w=w_0$

- Without damping ($\gamma = 0$), $\omega = \omega_0$.

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$$\cancel{+t(-\omega_0^2 A\cos(\omega_0 t) - \omega_0^2 B\sin(\omega_0 t))}$$

$$A = 0$$

$$B = \frac{F_0}{2\omega_0 m} = \frac{F_0}{2\sqrt{km}}$$

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$$x'' + \omega_0^2 x = \frac{F_0}{m}\cos(\omega_0 t) \qquad \omega_0 = \sqrt{\frac{k}{m}}$$

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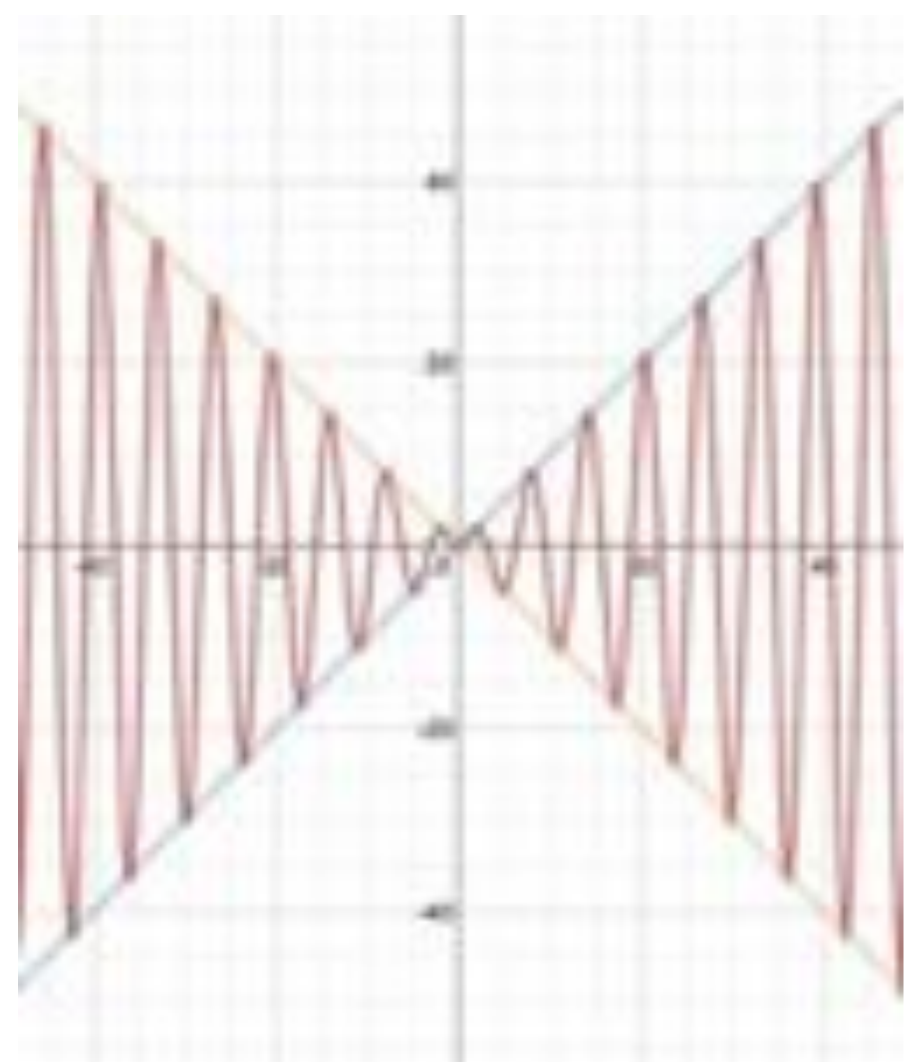
$$A = 0$$

$$B = \frac{F_0}{2\omega_0 m} = \frac{F_0}{2\sqrt{km}} \qquad x_p(t) = \frac{F_0}{2\sqrt{km}} t\sin(\omega_0 t)$$

Forced vibrations, no damping, w=w0

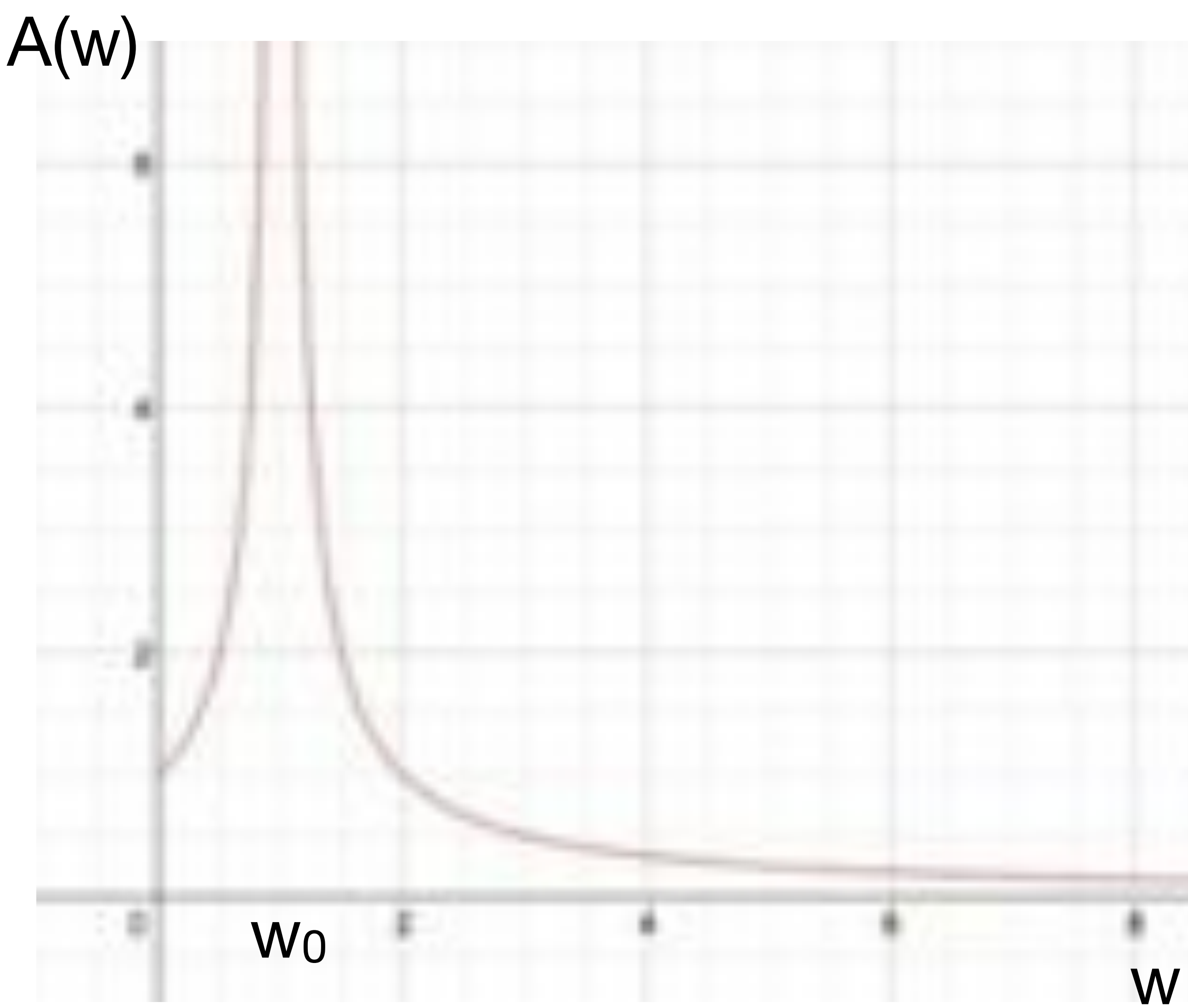
- Without damping ($\gamma = 0$), $\omega = \omega_0$.
  - Long term behaviour - x_p grows unbounded, swamping out x_h.

$$x_p(t) = \frac{F_0}{2\sqrt{km}} t \sin(\omega_0 t)$$



Forced vibrations, no damping, summary

- Plot of the amplitude of the particular solution as a function of ω.



- Calculated:

$$A = \frac{F_0}{m(\omega_0^2 - \omega^2)}$$

- Plotted with:

$$\frac{F_0}{m} = 1,\ \ w_0 = 1$$

$$A(\omega) = \frac{1}{|\omega_0^2 - \omega^2|}$$

- Recall that for $\omega = \omega_0$, the amplitude grows without bound.

Forced vibrations, with damping

$$mx'' + \gamma x' + kx = F_0 \cos\omega t$$

$$x'' + cx' + \omega_0^2 x = \frac{F_0}{m}\cos\omega t$$

No conflict with $x_h(t)$!

$$x_p = A\cos\omega t + B\sin\omega t$$

$$x_p' = -\omega A\sin\omega t + \omega B\cos\omega t$$

$$x_p'' = -\omega^2 A\cos\omega t - \omega^2 B\sin\omega t$$

$$-\omega^2 A\cos\omega t - \omega^2 B\sin\omega t + c(-\omega A\sin\omega t + \omega B\cos\omega t) + \omega_0^2(A\cos\omega t + B\sin\omega t) = \frac{F_0}{m}\cos\omega t$$

$$\underbrace{(-\omega^2 A + c\omega B + \omega_0^2 A)}_{F_0/m}\cos\omega t + \underbrace{(-\omega^2 B - c\omega A + \omega_0^2 B)}_{0}\sin\omega t = \frac{F_0}{m}\cos\omega t$$

$$A = \frac{F_0}{m}\,\frac{\omega_0^2 - \omega^2}{(c\omega)^2 + (\omega_0^2 - \omega^2)}$$

$$B = \frac{F_0}{m}\,\frac{c\omega}{(c\omega)^2 + (\omega_0^2 - \omega^2)}$$

$$x_p(t) = \frac{F_0}{m}\cdot\frac{1}{\sqrt{(c\omega)^2 + (\omega_0^2 - \omega^2)^2}}\left(\frac{(\omega_0^2 - \omega^2)}{\sqrt{(c\omega)^2 + (\omega_0^2 - \omega^2)^2}}\cos\omega t + \frac{c\omega}{\sqrt{(c\omega)^2 + (\omega_0^2 - \omega^2)^2}}\sin\omega t\right)$$

Forced vibrations, with damping