Full Name: ______________________
Student Number: ______________________

Math 103-201 Midterm Test 2 March 17, 2006

1. Place your answer to each question in the box. Your answer will be marked right or wrong (work will not be considered for this section).

   (a) Calculate $\int x\cos(x^2)\ dx$.

   $\boxed{\frac{1}{2}\sin(x^2) + C}$

   Solution
   Substituting $u = x^2, du = 2xdx$ gives

   $$\int x\cos(x^2)\ dx = \frac{1}{2}\int \cos u\ du = \frac{1}{2}\sin u + C = \frac{1}{2}\sin(x^2) + C.$$

   (b) What is probability that the sum of two rolls of a die is an odd number?

   $\boxed{\frac{1}{2}}$

   Solution

   $$P(\text{sum is odd}) = P(\text{roll 1 is even and roll 2 is odd}) + P(\text{roll 1 is odd and roll 2 is even})$$
   $$= \frac{1}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{2}$$

   (c) Calculate $\int_{-1}^{1}\sqrt{1+x^2}\ dx$.

   $\boxed{\sqrt{2} + \ln(\sqrt{2}+1)}$

   Solution
   Let $x = \tan\theta,\quad dx = \sec^2\theta d\theta$.

   $$\int_{-1}^{1}\sqrt{1+x^2}\ dx = \int_{-\pi/4}^{\pi/4}\sec^3\theta d\theta = 2\int_0^{\pi/4}\sec^3\theta d\theta = \left(\sec\theta\tan\theta + \ln|\sec\theta+\tan\theta|\right)\Big|_0^{\pi/4}$$
   $$= \sec\frac{\pi}{4}\tan\frac{\pi}{4} + \ln\left|\sec\frac{\pi}{4} + \tan\frac{\pi}{4}\right| - \sec 0\tan 0 + \ln|\sec 0 + \tan 0|$$
   $$= \sqrt{2}\cdot 1 + \ln|\sqrt{2}+1| - 1\cdot 0 + \ln|1+0| = \sqrt{2} + \ln(\sqrt{2}+1).$$

   Note that there are several different ways of expressing the answer depending on how you evaluate the integral.

   (d) If the probability of rain on any day in February was 0.6, what is the probability of it having rained on a total of exactly 26 days during that month? Reminder: February had 28 days this year. You do not have to reduce the expression you get.

   $\boxed{378\cdot(0.6)^{26}(0.4)^2}$

   Solution

   $$C(28,26)(0.6)^{26}(0.4)^2 = \frac{28!}{26!2!}(0.6)^{26}(0.4)^2 = 27\cdot 14\cdot(0.6)^{26}(0.4)^2$$

(e) Calculate $\int \frac{1}{x^2+6x+10}\, dx$.

$\boxed{\arctan(x+3)+C}$

Solution

$$\int \frac{1}{x^2+6x+10}\, dx = \int \frac{1}{(x+3)^2+1}\, dx = \arctan(x+3)+C.$$

(f) How many distinct words (potentially meaningless) can be made by rearranging the letters of the word "anagram" ? (e.g. granama, naamarg ...)

$\boxed{840}$

Solution

There are 7! ways of rearranging the letters. But because the a's are indistinguishable, each one of these 7! words come in indistinguishable groups of size 3! so there are really only $\frac{7!}{3!} = 840$ distinct words.

(g) Calculate $\int_{\pi/4}^{\pi/2} \csc^4(x)\, dx$.

$\boxed{\frac{4}{3}}$

Solution

$$\int_{\pi/4}^{\pi/2} \csc^4(x)\, dx = \int_{\pi/4}^{\pi/2} (1+\cot^2 x)\csc^2(x)\, dx.$$

Using the substitution $u = \cot x,\ du = -\csc^2 x\, dx$, this becomes:

$$-\int_1^0 (1+u^2)\, du = \int_0^1 (1+u^2)\, du = \left(u + \frac{1}{3}u^3\right)\Big|_0^1 = \frac{4}{3}.$$

(h) Consider a game in which you win if two dice, when rolled, show a total of 2. Each time you win you get \$35 dollars and each time you lose, you lose \$1. If you play 108 times, overall, how much do expect to win? (A loss should be denoted by a negative number.)

$\boxed{\$\ 0}$

Solution

$n = 108, p = \frac{1}{36}$.
Expected wins $= np = 108\frac{1}{36} = 3$.
Expected losses $= 105$.
Expected earnings $= 3 \cdot 35 - 1 \cdot 105 = 105 - 105 = 0$.

2. A rod with ends at $x = 0$ cm and $x = 1$ cm has a (linear) density given by $\frac{1}{1+x^2}$ grams per cm.

   (a) Find the total mass of the rod.

   (b) At what point should the rod be cut so that the two resulting pieces have equal weight?

   (c) At what point is the center of mass of the rod?

Solutions

(a)

$$M = \int_0^1 \frac{1}{1+x^2}\, dx = \arctan x|_0^1 = \frac{\pi}{4}$$

$$\boxed{M = \tfrac{\pi}{4} \text{ grams.}}$$

(b) To find the point that splits the mass in two (median point), solve the following equation for x_{half}:

$$\frac{\pi}{8} = \int_0^{x_{half}} \frac{1}{1+x^2}\, dx = \arctan x|_0^{x_{half}} = \arctan(x_{half}).$$

This can be rewritten as

$$x_{half} = \frac{\sin(\frac{\pi}{8})}{\cos(\frac{\pi}{8})} = \frac{\sqrt{\frac{1-\cos(\frac{\pi}{4})}{2}}}{\sqrt{\frac{1+\cos(\frac{\pi}{4})}{2}}} = \frac{\sqrt{1-1/\sqrt{2}}}{\sqrt{1+1/\sqrt{2}}} = \sqrt{\frac{\sqrt{2}-1}{\sqrt{2}+1}}.$$

$$\boxed{x_{half} = \sqrt{\tfrac{\sqrt{2}-1}{\sqrt{2}+1}} \text{ cm}}$$

(c) Using the substitution $u = 1 + x^2,\ du = 2xdx$ simplifies the expression for the center of mass:

$$x_{center} = \frac{\int_0^1 x\frac{1}{1+x^2}\, dx}{\frac{\pi}{4}} = \frac{4}{\pi}\int_0^1 x\frac{1}{1+x^2}\, dx = \frac{2}{\pi}\int_1^2 \frac{1}{u}\, du$$

$$= \frac{2}{\pi}\,(\ln u)|_1^2 = \frac{2}{\pi}\ln 2$$

$$\boxed{x_{center} = \tfrac{2}{\pi}\ln 2 \text{ cm}}$$

3. A child is playing a game in which 5 coloured beads must be hung vertically on a string, one above the other. Each bead is a different colour. The child does not pay attention to the colours while placing them on the string. Find the probability of finding

   (a) the blue bead on the bottom and the green bead at the top,

   (b) the orange bead second from the bottom or the blue bead at the top,

   (c) the green and purple beads not next to each other,

   (d) the red bead above both the green and blue beads.

   Solutions

   (a) $P(\text{B on bottom and G on top}) = P(\text{B on bottom})P(\text{G on top given B on bottom}) = \frac{1}{5} \cdot \frac{1}{4} = \boxed{\frac{1}{20}.}$

   (b) $P(\text{O second or B on top}) = P(\text{O second}) + P(\text{B on top}) - P(\text{O second and B on top}) = \frac{1}{5} + \frac{1}{5} - \frac{1}{20} = \boxed{\frac{7}{20}.}$

   (c) There are 8 arrangements in which green and purple touch and for each one of these, there are 3! ways of arranging the other 3 beads. With a total of 5! ways of arranging the beads, the probability of green and purple touching is $\frac{8 \cdot 3!}{5!} = \frac{2}{5}$. So the probability of green and blue not being adjacent is $1 - \frac{2}{5} = \boxed{\frac{3}{5}.}$

   (d) The child has to unwittingly and randomly choose one of the red, green and blue beads to be highest among the three. So there is a $\frac{1}{3}$ chance of choosing the red to be above the green and blue. If you don't like that explanation, you can decompose it as follows (many equivalent arguments can be made):

$$
\begin{aligned}
& P(\text{R is left of G and B}) \\
&= P(\text{R is at the very top}) + P(\text{R, G and B are not at the very top and R is second down}) \\
&\quad + P(\text{R, G and B are in neither of the top two spots and R is third down}) \\
&= \frac{1}{5} + \frac{2}{5} \cdot \frac{1}{4} + \frac{2}{5} \cdot \frac{1}{4} \cdot \frac{1}{3} \\
&\boxed{= \frac{1}{3}}
\end{aligned}
$$