Mathematics 103 Section 203 Full Name: ____________________
Instructor: L. Keshet Student Number: ____________________

Midterm Test Feb 10, 2005

Instructions: There are **5 pages** in this test (including this cover page).

1. Ensure that your *full* name and student number appears on this page.
2. No calculators, books, or notes, or electronic devices of any kind are permitted.
3. Show all your work. Answers not supported by calculations or reasoning will not receive credit. Messy work will not be graded.
4. Exposing your test paper, copying from another student's paper, or sharing information about this test constitutes academic dishonesty. Such behaviour may jeopardize your grade on this test, in this course, and your standing at this university.
5. Five minutes before the end of the test period you will be given a verbal notice. After that time, you must remain seated until all test papers have been collected.
6. When the test period is over, you will be instructed to put away writing implements. Put away all pens and pencils at this point. Continuing to write past this instruction will be considered as cheating.
7. Please remain seated and pass your test paper down the row to the nearest indicated aisle. Once all the test papers have been collected, you are free to leave.

Question	Grade	Value
1		10
2		10
3		10
4		12
5		8
Total		50

I have read and understood the instructions and agree to abide by them.

Signed: ____________________

Some Formulae:

$$\sum_{k=1}^{N} k^2 = \frac{(2N+1)N(N+1)}{6},$$

$$\sum_{k=1}^{N} k^3 = \left(\frac{N(N+1)}{2}\right)^2.$$

Problem 1: This problem has several unrelated parts. Carefully check your work. Give units where applicable. This problem will not be awarded partial credit for incorrect answers.

(a) Compute the following integrals (by any method)

$$\int_0^1 e^{-2t}\, dt = \rule{10cm}{0.4pt}$$

$$\int_0^2 |x-1|\, dx = \rule{10cm}{0.4pt}$$

Solution:

$$\int_0^1 e^{-2t}\, dt = (-1/2)e^{-2t}\Big|_0^1 = (1/2)(1-e^{-2})$$

$$\int_0^2 |x-1|\, dx = \int_0^1 -(x-1)\, dx + \int_1^2 (x-1)dx = 2\int_1^2 (x-1)dx = 1$$

(b) The rate of growth of an embryo is $f(t) = A^2 - t^2$ (gm/day) for $0 \le t \le A$ days, where $A > 0$ is a constant. Answer the following (in terms of A).

The total growth over this period is ____

The average growth rate over this period is ____________________________________

Solution:

The total growth is $M = \int_0^A f(t)dt = \int_0^A (A^2-t^2)dt = \left(A^2t - \frac{t^3}{3}\right)\Big|_0^A = A^3(1-\frac{1}{3}) = \frac{2}{3}A^3$ gm

The average growth rate is $\bar{f} = \frac{1}{A}\int_0^A f(t)dt = \frac{1}{A}\frac{2}{3}A^3 = \frac{2}{3}A^2$ gm/day.

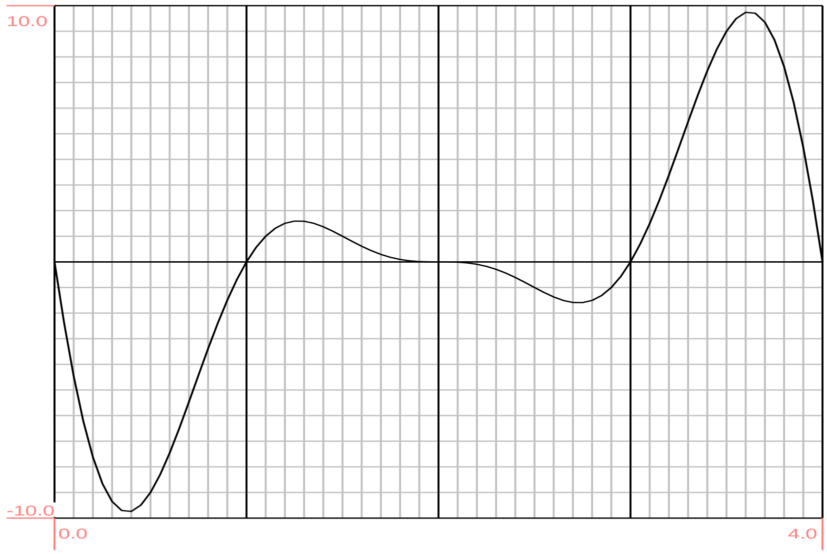
(c) Compute the following sum: $\sum_{n=1}^{4}(n-1)^3 =$ ____________________________________

$$\sum_{n=1}^{4}(n-1)^3 = 0^3 + 1^3 + 2^3 + 3^3 = 1 + 8 + 27 = 36$$

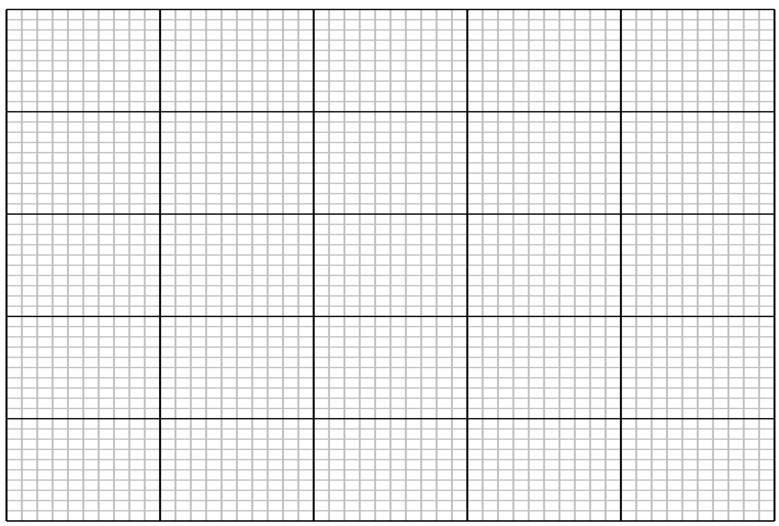
Problem 2: Shown in Figure 1 (on the next page) is the acceleration of an elevator. Sketch the corresponding velocity and position of the elevator during the same time interval on the grids provided on that page. Assume that initially $v(0) = 0$ and $x(0) = 0$. (You will get points for overall shapes of your graphs. The more accurate the graph, the more points you can earn.)

Solution: See graphs. Note that the acceleration graph has symmetry: the area below and above the x axis on that graph is the same for the interval $0 \le t \le 4$. This means that the velocity at $t = 4$ will be the same as it was at $t = 0$. Also note that v has horizontal tangents at all points where $a(t) = 0$. Similarly, $x(t)$ has horizontal tangents at places where $v(t) = 0$, i.e. at $t = 0, 4$.

Acceleration



Velocity



Position

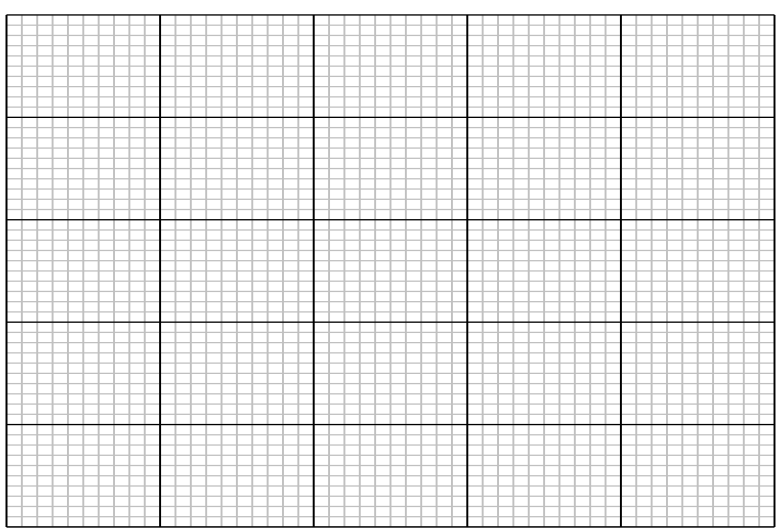


Figure 1: Figure for Problem 2

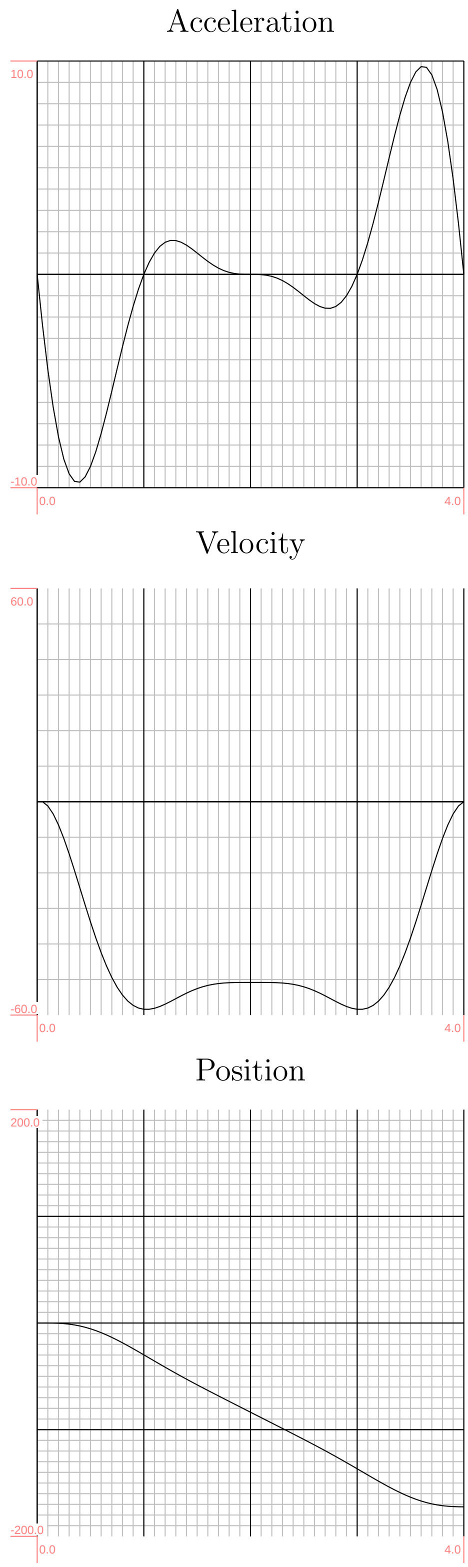


Figure 2: Figure for solution to Problem 2

Problem 3: Consider the two functions $y = f(x) = 6x - x^3$ and $y = g(x) = x^2$. These functions intersect at some point p in the positive quadrant. Find p and then compute the area between the two graphs for $0 \le x \le p$. Your final answer should be a number, written as a (simplified, i.e., common denominator) fraction.

Solution to Problem 3: For the functions $y = f(x) = 6x - x^3$ and $y = g(x) = x^2$, the curves intersect when $6x - x^3 = x^2$, i.e. $x^3 + x^2 - 6x = 0$. This can be simplified to $x(x^2 + x - 6) = 0$ or further factored into $x(x+3)(x-2) = 0$. For $x \ge 0$ the desired interval is $0 \le x \le 2$. By plugging in any point, e.g. $x = 1$ it can be seen that the cubic curve is higher than the quadratic curve. The integral to compute is:

$$I = \int_0^2 (6x - x^3 - x^2)dx = \left(6\frac{x^2}{2} - \frac{x^4}{4} - \frac{x^3}{3}\right)\Big|_0^2 = \left(12 - 4 - \frac{8}{3}\right) = \frac{16}{3}$$

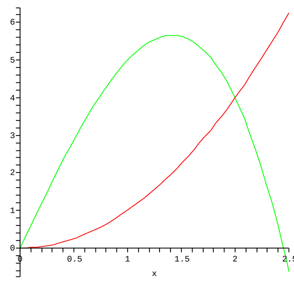


Figure 3: Figure for solution to Problem 3

Problem 4: Use the "disk method" to find the volume of the ellipsoid obtained by revolving the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$

(a) around the x axis,

(b) around the y axis.

(c) The general equation of an ellipse is $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. What relationship between a and b would lead to the same volumes computed in these two distinct revolutions? (Do not redo calculation. State the relationship and support it with a simple 1 sentence answer.)

Solution to Problem 4:

For the ellipsoid $\frac{x^2}{4}+\frac{y^2}{9}=1$:

(a) about the x axis, the thickness of disks is along the x axis. The ellipse intersects the x axis at $y=0$, i.e. $x=\pm 2$. We must express the radius as $r=y=f(x)$. Then $r^2=y^2=9(1-x^2/4)$ so the integral (using the disk method) is

$$V=\pi\int_{-2}^{2}9(1-\frac{x^2}{4})dx=18\pi\int_{0}^{2}(1-\frac{x^2}{4})dx=18\pi(x-\frac{x^3}{12})\Big|_0^2=18\pi(2-\frac{8}{12})=18\pi(\frac{16}{12})=24\pi$$

(b) about the y axis the thickness of disks is along the y axis. The ellipse intersects the 0 axis at $x=0$, i.e. $y=\pm 3$. We must express the radius as $r=x=f(y)$. Then $r^2=x^2=4(1-y^2/9)$ so the integral (using the disk method) is

$$V=\pi\int_{-3}^{3}4(1-\frac{y^2}{9})dy=8\pi\int_{0}^{3}(1-\frac{y^2}{9})dy=8\pi(y-\frac{y^3}{27})\Big|_0^3=8\pi(3-1)=16\pi$$

(c) The two volumes are equal in the symmetric case $a=b$.

Problem 5: Shown in Figure 4 are the rates of flow of water into a tank over 12 hours. Answer the following questions (to the nearest half-hour).

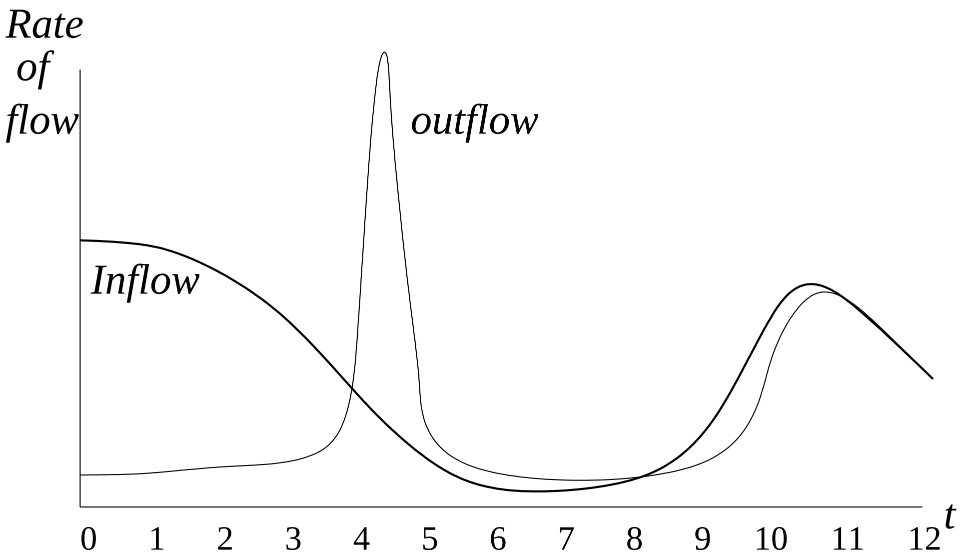


Figure 4: Figure for Problem 5

(a) At what time is the level of water in the tank highest?

(b) At what time is the level of water in the tank lowest?

(c) At what time is the water level changing at the fastest rate?

(d) Over what time *interval* is the water level not changing?

Solution:

(a) The level of water in the tank is highest at about $t \approx 4$, since the inflow has been larger than the outflow for an extended period of time. (The answer $t \approx 11$ is not acceptable, since the area between the outflow and inflow curves over $4 \leq t \leq 8$ is larger than the area between the inflow and outflow curves over $8 \leq t \leq 11$.)

(b) The level of water in the tank is lowest at $t \approx 0$. There is a local minimum at $t = 8$, but this is not as low as the level at $t = 0$ because the area between the curves after $t = 4$ (net loss) is smaller than the area between the curves before $t = 4$ (net gain).

(c) The water level is changing at the fastest rate when the difference between the in and out flows is largest. This occurs at $t \approx 4.5$. (Answers in the range $4 \leq t \leq 5$ is acceptable.)

(d) The water level is not changing over the time interval $11 \leq t \leq 12$, since the inflow and outflow rates are the same.