Monday Oct 16

Last time: defined the derivative of a function at a point

Definition: Given a function f continuous at $x=a$, the derivative of f <u>at $x=a$</u> is defined as

$$\lim_{h \to 0} \frac{f(a+h)-f(a)}{h} = f'(a)$$

Today: we define the derivative function.

Interpretions

1) $f'(a)$ = rate of change of f at $x=a$

Note: if x = time, $f(x)$ = position

$f'(a)$ = <u>velocity</u> instantaneous at $x=a$

2) $f'(a)$ = slope of the line tangent to the graph of f at $(a, f(a))$

<u>one</u> definition $\longrightarrow$ <u>two</u> interpretations

Example 1: A field is sprayed with insecticide. After 1 hour, the bugs start to die.
The number of bugs $b(t)$ at time t after spraying is $\boxed{b(t) = \frac{1}{t}}$ for $t>$
At what rate is the number of bugs decreasing at $t=2$?

Ans: rate of change of $b(t)$ at $t=2$ is $b'(2)$

A. $b'(2) = \lim_{h \to 0} \dfrac{\frac{1}{2} + h - \frac{1}{2}}{h}$ NO

$f(a) + h - f(a)$

B. $b'(2) = \lim_{h \to 0} \dfrac{\frac{1}{2+h} - \frac{1}{2}}{h}$ $\leftarrow$ careful $f(a+h) \neq f(a) + h$

$f(a+h)$

C. $b'(2) = \lim_{h \to 2} \dfrac{\frac{1}{2+h} - \frac{1}{2}}{h}$ NO

?

D. $b'(2) = \lim_{h \to 2} \dfrac{\frac{1}{2} + h - \frac{1}{2}}{h}$ NO

?

E. $b'(2) = \lim_{h \to 0} \dfrac{\frac{1}{t+h} - \frac{1}{t}}{h}$ NO.

[$f(a+h)$ is a function composition]

or $f(a+h)$ is function ~~evaluation~~ evaluation at $a+h$

$$b'(2) = \lim_{h \to 0} \frac{\frac{1}{2+h} - \frac{1}{2}}{h} =$$

$$= \lim_{h \to 0} \frac{\left(\frac{2-(2+h)}{(2+h)(2)}\right)}{h} =$$

$$= \lim_{h \to 0} \frac{2-2-h}{(2+h)(2)} \cdot \frac{1}{h}$$

$$= \lim_{h \to 0} \frac{-\cancel{h}}{(2+h)\cdot 2 \cdot \cancel{h}} = \lim_{h \to 0} \frac{-1}{2(2+h)} =$$

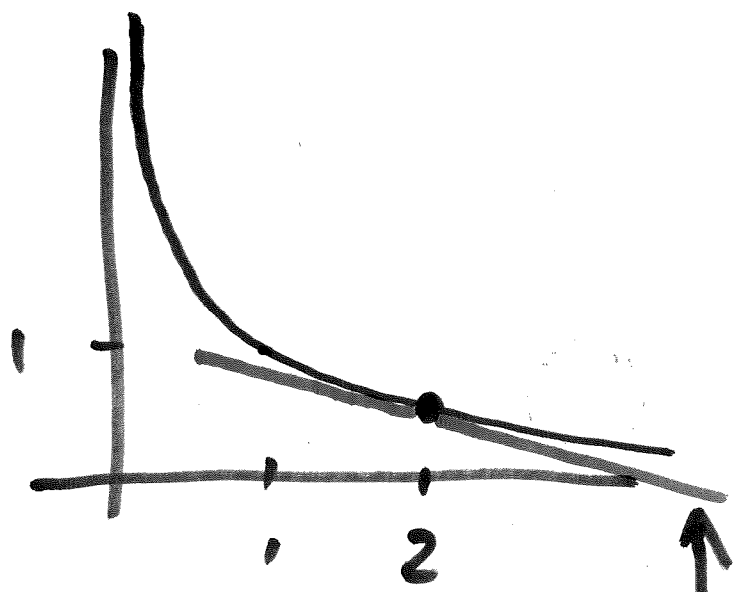
$$= -\frac{1}{2 \cdot 2} = -\frac{1}{4}$$

negative rate of change makes sense because the bugs are dying so $b(t)$ is decreasing at $t = 2$

In general, if a quantity $f(t)$ is

- <u>decreasing</u> at $t=a$, we expect $\underline{f'(a) < 0}$
- <u>increasing</u> at $t=a$, we expect $\underline{f'(a) > 0}$

Graphically,



$b(t) = \frac{1}{t} \quad t > 0$

we know

$b'(2) = -\frac{1}{4}$

slope of tangent at $(2, \frac{1}{2})$

Example 2:

Find the equation of the tangent line to $f(x) = \frac{1}{x}$ at $x = 2$.

slope $m = f'(2) = -\frac{1}{4}$ (from previous example)

point of tangency (lies on both the line and the curve)

$P(2, \frac{1}{2})$

point-slope equation of a line : $y - y_1 = m(x - x_1)$

So we have

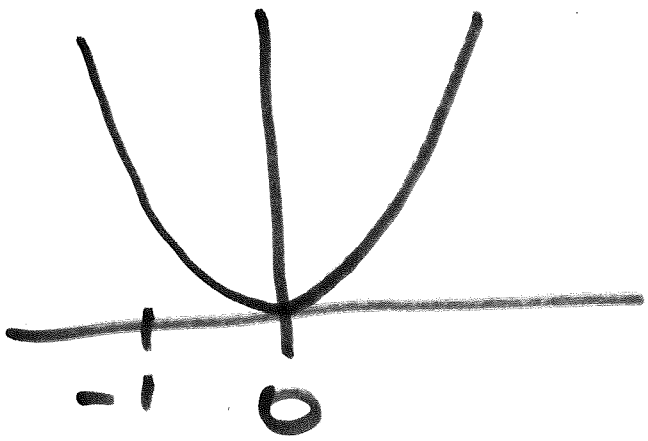
$$y - \frac{1}{2} = -\frac{1}{4}(x - 2)$$

$$y = -\frac{1}{4}x + \frac{1}{2} + \frac{1}{2}$$

$$\boxed{y = -\frac{1}{4}x + 1}$$

Derivative Function

$y = x^2$



we expect the slope of tangent to be negative if $x<0$ and positive if $x>0$

let's check:

$$f'(-1) = \lim_{h \to 0} \frac{(-1+h)^2 - (-1)^2}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{1} - 2h + h^2 \cancel{-1}}{h} =$$

$$= \lim_{h \to 0} \frac{\cancel{h}(-2+h)}{\cancel{h}} = -2$$

$$f'(0) = \lim_{h \to 0} \frac{(0+h)^2 - (0)^2}{h} = \lim_{h \to 0} \frac{h^2}{h} = 0$$

tangent line is horizontal

$f'(1) = \ldots$

$f'(2) = \ldots$

x	$f'(x)$
-1	-2
0	0
1	2
2	4
$\vdots$	$\vdots$

we can build a function $f'(x)$ using these values

In general, ~~given a fun~~ the derivative function $f'(x)$ is defined as

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Wed - Oct 18

Last time:

*) derivative at a point

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} = \text{a number}$$

$\downarrow$

represents $\begin{cases} \text{slope of tangent at } x=a \\ \text{rate of change at } x=a \end{cases}$

*) derivative function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \text{a function of } x$$

if evaluated at $x=a$, $f'(a)$

<u>Today</u>: how to $\left\{\begin{matrix}\text{find} \\ \text{compute}\end{matrix}\right\}$ the derivative function.

Ex 1: Find the derivative of $\boxed{f(x)=x^2}$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^2-x^2}{h} = \text{"simplify"}$$

$$=\lim_{h\to 0}\frac{\cancel{x^2}+2xh+h^2-\cancel{x^2}}{h}=$$

$$=\lim_{h\to 0}\frac{\cancel{h}(2x+h)}{\cancel{h}}=\lim_{h\to 0}(2x+h)=\boxed{2x}$$

h is the variable inside the limit.
x is a fixed number here

so the derivative of $f(x)=x^2$ is $f'(x)=2x$

Ex 2: Find the derivative of $\boxed{f(x)=x^3}$

$$f'(x)=\lim_{h\to 0}\frac{(x+h)^3-x^3}{h}=\lim_{h\to 0}\frac{\cancel{x^3}+3x^2h+3xh^2+h^3-\cancel{x^3}}{h}$$

$$=\lim_{h\to 0}\frac{3x^2\cancel{h}+3xh^{\cancel{2}}+h^{\cancel{3}\,2}}{\cancel{h}}=\lim_{h\to 0}3x^2+\underbrace{3xh}_{\to 0}+\underbrace{h^2}_{\to 0}$$

$$=\boxed{3x^2}$$

there seems to be a pattern here

$f(x)$	$f'(x)$
x^2	$2x$
x^3	$3x^2$
x^4	$4x^3$
x^5	$5x^4$
$\vdots$	$\vdots$
x^n	nx^{n-1}

Power Rule

The derivative of x^n is nx^{n-1}

$(x^n)' = nx^{n-1}$ for any n

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Ex: if $f(x) = \sqrt{x} = x^{1/2}$

$$f'(x) = \frac{1}{2}x^{\frac{1}{2}-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}$$

Ex: if $f(x) = \frac{1}{x}$, $f'(x) = ?$

A. x^{-2}

B. $\boxed{-x^{-2}}$

C. $x^0 = 1$

D. have no idea. ← you need to review <u>before Friday</u>

Recall $\frac{1}{x^n} = x^{-n}$ so $\frac{1}{x} = x^{-1}$

What if $f(x) = 3$? $f(x) = 3 \cdot x^0$

we expect $f'(x) = 0$.

let's check.

$$f'(x) = \lim_{h \to 0} \frac{\overset{f(x+h)}{3} - \overset{f(x)}{3}}{h} = \lim_{h \to 0} \frac{0}{h} = 0$$

What if $f(x) = 4x^5$?

In general, if $f(x) = c\,g(x)$ where c is a constant, then

$$f'(x) = \lim_{h \to 0} \frac{\overset{f(x+h)}{c\,g(x+h)} - \overset{f(x)}{c\,g(x)}}{h} =$$

$$\lim_{h \to 0} \frac{c\,(g(x+h) - g(x))}{h} =$$

recall $\lim_{x \to a} k\,F(x) = k \lim_{x \to a} F(x)$

so $c \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{g'(x)} = c\,g'(x)$

Constant-multiple rule

If $f(x) = c\,g(x)$ for some constant c,

$$f'(x) = c\,g'(x)$$

$$\frac{d}{dx} f(x) = c \frac{d}{dx} g(x)$$

$$(c f(x))' = (c\,g(x))' = c\,g'(x)$$

Ex: if $f(x) = 4x^5$, $f'(x) = 4 \cdot (x^5)'$

so $f'(x) = 4 \cdot 5x^{5-1} = 20x^4$

What if $f(x) = 3x^2 + \frac{1}{2}x^5$?

In general, if $h(x) = f(x) + g(x)$

then $h'(x) = \lim_{h \to 0} \dfrac{\overbrace{(f(x+h) + g(x+h))}^{h(x+h)} - \overbrace{(f(x) + g(x))}^{h(x)}}{h}$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x) + g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + \frac{g(x+h) - g(x)}{h} =$$

$$= \underbrace{\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}_{f'} + \underbrace{\lim_{h \to 0} \frac{g(x+h) - g(x)}{h}}_{g'}$$

<u>Sum Rule</u> If $f(x) \pm g(x) = F(x)$

then $F'(x) = f'(x) \pm g'(x)$

$$\frac{d}{dx}\left(f(x) \pm g(x)\right) = \frac{df}{dx} \pm \frac{dg}{dx}$$

Ex $f(x) = 4x^{100} - \frac{1}{2}\frac{1}{\sqrt[3]{x^2}}$

Recall $\sqrt[3]{x^2} = (x^2)^{1/3} = x^{2/3}$

So $f(x) = 4x^{100} - \frac{1}{2}x^{-2/3}$

$f'(x) = 4 \cdot 100\, x^{100-1} - \frac{1}{2}\left(-\frac{2}{3}\right)x^{-\frac{2}{3}-1}$

$f'(x) = 400\, x^{99} + \frac{1}{3}x^{-5/3}$

Summary of important concepts and formulae we have seen so far:

1) Derivative function (definition)

$$f'(x) = \frac{df}{dx} = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

2) interpretation of derivative at a point

At $x=a$, $f'(a)$ represents $\begin{cases} \text{slope of tangen} \\ \text{rate of change of } f. \end{cases}$

3) Differentiation Rules

•) Power rule : $(x^n)' = nx^{n-1}$ or $\frac{d}{dx}(x^n) = nx^{n-1}$

•) Constant rule : $(c)' = 0$ or $\frac{d}{dx}(c) = 0$

constant

•) Constant multiplier rule : $(c f(x))' = c f'(x)$

$$\frac{d}{dx}[c f(x)] = c \frac{d}{dx} f(x)$$

•) Sum/difference rule :

$$(f(x) + g(x))' = f'(x) + g'(x)$$

$$y' = (3x^{10})' - \left(\frac{1}{3}x^3\right)'$$

$$= 3\cdot(x^{10})' - \frac{1}{3}(x^3)'$$

$$= 3\cdot 10x^{10-1} - \frac{1}{3}3\cdot x^{3-1} =$$

$$= 30x^9 - x^2$$

$$y = 2\sqrt[3]{x^2} - 5\sqrt{x^3}$$

$$y = 2\cdot(x^2)^{1/3} - 5(x^3)^{1/2}$$

$$y = 2x^{2/3} - 5x^{3/2}$$

$$y' = 2\cdot\frac{2}{3}x^{\frac{2}{3}-1} - 5\cdot\frac{3}{2}x^{\frac{3}{2}-1}$$

$$= \frac{4}{3}x^{-\frac{1}{3}} - \frac{15}{2}x^{\frac{1}{2}}$$ (leave it as i

$$= \frac{4}{3}\frac{1}{\sqrt[3]{x}} - \frac{15}{2}\sqrt{x}$$

Announcements (Fri - Oct 20)

1) Midterm on October 24, 6 PM (Tuesday)

2) No workshops next week

3) No written assignment due next week

4) WebWork assignments are due as per usual schedule.

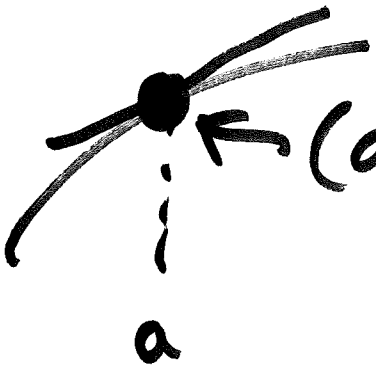
Today (Friday October 20)

Practise solving problems using derivatives

Ex 1: Find the equation of the tangent line to $f(x) = 10x^{2/3} - 8x^5$ at $x = 1$.

Strategy: to find the equation of a line we need
- slope m
- point (x_0, y_0)

we know $f'(a)$ = slope of tangent line at $(a, f(a))$



$(a, f(a))$ point of tangency

↓

this point lies on both the line and the curve $f(x)$.

<u>Solution</u> :

compute slope of tangent as $m = f'(1)$.

$$f'(x) = 10\,\frac{2}{3}\,x^{\frac{2}{3}-1} - 8.5\,x^{5-1}$$

$$f'(x) = \frac{20}{3}\,x^{-1/3} - 40\,x^{4}$$

$$f'(1) = \frac{20}{3}\,1^{-1/3} - 40\cdot 1 = \frac{20}{3} - 40$$

$$= -\frac{100}{3} = m$$

point of tangency ~~P~~: ~~$x_p = 1$~~

$P(x_p, y_p)$: $x_p = 1$

$y_p = 10 \cdot 1^{2/3} - 8 \cdot 1^5 = 2$

point-slope form : $y - y_0 = m(x - x_0)$

here $m = -\frac{100}{3}$ and $(x_0, y_0) = (1, 2)$

so eq. of the tangent line at $x=1$

is $y - 2 = -\frac{100}{3}(x-1)$

$$y = 2 - \frac{100}{3}(x-1)$$

Ex 2: Find the points of the curve $f(x) = \frac{2}{3}x^3 - 4x^2 + 4x + 10$ where the tangent line is horizontal.

Strategy:

horizontal line has slope $m=0$

so we set $f'(x)=0$

and solve for x.

<u>Solution</u>:

$$f'(x) = \frac{2}{\cancel{3}} \cdot \cancel{3}x^2 - 4 \cdot 2x^1 + 4 \cdot x^0 + 0$$

$$f'(x) = 2x^2 - 8x + 4$$

for tangent line to be horizontal

we need $f'(x) = 0$

$$2x^2 - 8x + 4 = 0$$

$$x^2 - 4x + 2 = 0$$

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 1 \cdot 2}}{2} = \frac{4 \pm \overset{2\sqrt{2}}{\sqrt{8}}}{2}$$

$$x = \frac{\cancel{2}(2 \pm \sqrt{2})}{\cancel{2}} = 2 \pm \sqrt{2}$$

$$x = \frac{4 \pm \sqrt{8}}{2} = \frac{4 \pm 2\sqrt{2}}{2} =$$

$$= \frac{\cancel{2}(2 \pm \sqrt{2})}{\cancel{2}} = 2 \pm \sqrt{2}$$

There are two points ~~$x = 2 \pm \sqrt{}$~~

$x_1 = 2 + \sqrt{2}$, $f(x_1) = y$

and $x = 2 - \sqrt{2}$, $f(x_2) = y$

where the tangent line is horizontal.