Today:  
- graphing $f'$ from $f$
- computing derivatives
- solving tangent lines problems

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Last time

Defn: Derivative at a point

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

$f'(a)$ is a number representing:

1) slope of line tangent to curve at point $(a, f(a))$

2) rate of change of $f$ at $x=a$.

(If $x$ is time and $f(x)$ is position, $f'(a)$ is velocity at time $x=a$)
Definition: Derivative Function

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

so if \( x = a \), \( f'(a) \) as above.

Notation: \( y = f(x) \)

Derivative of \( f \):

\( y', \ y'(x), \ f', \ f'(x), \)

\[ \frac{df}{dx}, \ \frac{d}{dx} (f) \]

\( \Rightarrow \) "d" \( f \) "d" \( x \).
Graphing $f'$ from $f$

**Example:**

$y = f(x)$

Decide where $f' = 0$, $f' > 0$, $f' < 0$

**Graph:**

$y = f'(x)$
Computing Derivatives

we know $f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

Computing limits is time consuming and challenging.

Shortcuts: Differentiation Rules.

1. Power Rule

$y = x^2$, $(x^2)' = 2x$

$y = x^3$, $(x^3)' = 3x^2$

$(x^4)' = 4x^3$

In general, for any $n$, $(x^n)' = nx^{n-1}$
(2) Sum and difference Rule

\[(f(x)+g(x))' = ?\]

\[(f(x)+g(x))' = \text{by def} = \lim_{h \to 0} \frac{[f(x+h)+g(x+h)]-[f(x)+g(x)]}{h} = \lim_{h \to 0} \frac{f(x+h)-f(x) + g(x+h)-g(x)}{h} = \lim_{h \to 0} \frac{f(x+h)-f(x)}{h} + \lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = f'(x) + g'(x)\]
ex: \( y = x^{10} \)
\[
\frac{dy}{dx} = 10x^{10-1} = 10x^9
\]
\[ y = x^{-3} \]
\[
\frac{dy}{dx} = -3x^{-3-1} = -3x^{-4}
\]

Aside: Recall \( x^{-n} = \frac{1}{x^n} \)

ex \(-3x^{-4} = -3\frac{1}{x^4} = -\frac{3}{x^4}\)

\[ y = x^{3/2} \]
\[
\frac{dy}{dx} = \frac{3}{2}x^{3/2-1} = \frac{3}{2}x^{1/2}
\]

Aside: Recall \( x^{\frac{1}{n}} = \sqrt[n]{x} = \sqrt[x]{x} \)

ex \( \frac{3}{2}x^{1/2} = \frac{3}{2}\sqrt{x} \)
The derivative of a sum = sum of the derivatives.

Example: \( y = x^{100} + x^{-\frac{1}{5}} \)

\[
y' = (x^{100})' + (x^{-\frac{1}{5}})'
\]

\[
= 100 x^{99} - \frac{1}{5} x^{-\frac{6}{5}}
= 100 x^{99} - \frac{1}{5} \frac{1}{\sqrt[5]{x^6}}
\]
3) Constant Multiple Rule

\[(cf(x))' = ? \text{ where } c \text{ is a constant.}\]

By def:

\[
\lim_\limits{h \to 0} \frac{cf(x+h) - cf(x)}{h} =
\]

\[
= \lim_\limits{h \to 0} c \frac{f(x+h) - f(x)}{h} = cf'(x)
\]

\[\boxed{(cf(x))' = cf'(x)}\]

ex: \[y = 3x^{-5}\]

\[y' = 3(x^{-5})' = 3 \cdot (-5)x^{-6}\]
\[ y' = -15x^{-6} = -\frac{15}{x^6} \]

\[ y = 3x^{10} - \frac{1}{3}x^3 \]

\[ y' = 3 \cdot 10x^9 - \frac{1}{3} \cdot 3x^2 \]

\[ = 30x^9 - x^2 \]

\[ y = 2\sqrt[3]{x^2} - 5\sqrt[3]{x^3}. \]

\[ y = 2x^{2/3} - 5x^{3/2} \]

\[ y' = 2 \cdot \frac{2}{3}x^{\frac{2}{3}-1} - 5 \cdot \frac{3}{2}x^{\frac{3}{2}-1} \]

\[ = \frac{4}{3}x^{-\frac{1}{3}} - \frac{15}{2}x^{1/2} = \]
\[ \frac{4}{3} \sqrt[3]{\frac{1}{x}} - \frac{15}{2} \sqrt{x}. \]
Today: use derivatives to solve
  • rate of change problems
  • tangent line problems

Last time: Differentiation Rules

1. power rule \((x^n)' = nx^{n-1}\)
2. sum or difference rule : \((f+g)' = f' + g'\)
3. constant multiple : \((cf)' = cf'\)

Ex: \(y = 3\sqrt[4]{x^2} - \frac{1}{2}\sqrt[3]{x^3}\)

\[y = 3x^{\frac{2}{4}} - \frac{1}{2}x^{\frac{3}{3}} = 3x^{\frac{1}{2}} - \frac{1}{2}x^{-\frac{3}{5}}\]

\[y' = (3x^{1/2})' - \left(\frac{1}{2}x^{-\frac{3}{5}}\right)' =\]

\[= 3 \left(x^{1/2}\right)' - \frac{1}{2} \left(x^{-3/5}\right)' =\]

\[= 3 \cdot \frac{1}{2} x^{\frac{1}{2} - 1} - \frac{1}{2} \left(-\frac{3}{5}\right) x^{-\frac{3}{5} - 1} =\]

\[\Rightarrow y' = \frac{3}{2} x^{-\frac{1}{2}} + \frac{3}{10} x^{-\frac{8}{5}}\]
Problems involving rates of change

1. Moving object, \( y(x) = -4x + \frac{1}{2}x^2 \) is position at time \( x \) (min).

Find velocity of obj. at time \( x = 3 \).

Velocity \( y'(x) = -4 \cdot 1x^{1-1} + \frac{1}{2} \cdot 2x^{2-1} \)

\[ = -4 + x \]

\[ = -4 + x \]

When \( x = 3 \), \( y'(3) = -4 + 3 = -1 \text{ m/min} \)

Derivative of a constant function

\( f(x) = K \) where \( K \) is a constant

\[ \begin{array}{c}
\text{K} \\
\hline
0 \\
\end{array} \]

Horizontal line of equation \( y = k \)

Rate of change of \( f = 0 \) everywhere

Let's verify it.
\[
f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{k - k}{h} = \lim_{h \to 0} \frac{0}{h} = 0.
\]

(2) Temperature in this room (degrees) at time \( t \) is \( T'(t) = 13 + \sqrt{2} \cdot t^2 \) (completely unrealistic!)
What is the rate of change of temperature at \( t = 9 \) (hours)?
\[
T'(t) = 0 + \sqrt{2} \cdot 2t
\]
\[
T'(9) = 2\sqrt{2} \cdot 9 = 18\sqrt{2} \text{ degree/hr.}
\]
3) Find the slope of tangent line to $y = 3 - 2x^2$ at $(1, 1)$.

Expect $m < 0$.

Slope = $m$, $m = y'(1)$

$y'(x) = -2 \cdot 2x = -4x$

$y'(1) = -4 \cdot 1 = -4 < 0$ !

4) Find the equation of a tangent line to $y = x^3 - 4x$ at the point of $x$-coordinate $x = 2$.

Point of tangency $(x_0, y_0)$ lies on both tangent and curve.
Strategy: slope of tangent

Slope \( m = y'(2) \)

Point of tangency \( (x_0, y_0) = (2, 0) \)

\[ y = x^3 - 4x \]

at \( x = 2 \)

\[ y = 2^3 - 4 \cdot 2 = 0 \]

Solution:

\[ y' = 3x^2 - 4 \]

\[ y'(2) = 3 \cdot 2^2 - 4 = 8 \]

\( m = 8, \ (2, 0), \ point-slope \ form \)

\[ y - y_0 = m (x - x_0) \]

\[ y - 0 = 8(x - 2) \]

\[ y = 8x - 16 \]
5. Find the intercepts (with x and y axes) of the line tangent to 

\[ y = 3 \sqrt{x} \]

at the point of x-coord. 

\[ x = 4 \]

Strategy: need eq. of the tangent

\[ m = y'(4) \]

Tangency point \( (x_0, y_0) = (4, 3\sqrt{4}) = (4, 4) \)

Tangent: 

\[ y - y_0 = m(x - x_0) \]

Intercepts: 

\[ x = 0, \ldots \]
\[ y = 0, \ldots \]
Solution \[ y = 3 \sqrt{x} = 3x^{\frac{1}{2}} \]

\[ y' = 3 \cdot \frac{1}{2} x^{\frac{1}{2}-1} = \frac{3}{2} x^{-\frac{1}{2}} \]

\[ y'(4) = \frac{3}{2} \cdot \frac{1}{4^{\frac{1}{2}}} = \frac{3}{2} \cdot \frac{1}{2} \cdot \frac{1}{\sqrt{4}} = \frac{3}{2} \cdot \frac{1}{2} = \frac{3}{4} \]

\[ m = \frac{3}{4} \]

\( (x_0, y_0) = (4, 6) \)

eq \text{ of tangent } \hspace{1cm} y - 6 = \frac{3}{4} (x - 4) \]

\[ y = \frac{3}{4} x - 3 + 6 \]

\[ y = \frac{3}{4} x + 3 \]

\[ \boxed{y = \frac{3}{4} x + 3} \]

x-intercept: \[ y = 0 \]

\[ \frac{3}{4} x + 3 = 0 \]

\[ \frac{3}{4} x = -3 \]

\[ x = -\frac{12}{4} = -4 \]
y-intercept:  
\[ x = 0 \]
\[ y = 3 \]

Point of tangency:  
\[ y = 3\sqrt{x} \]

(0, 3)
Find the equation of the lines that are tangent to $y = x^2$ and go through the point $(1, -2)$.

**Strategy**

Point of tangency is unknown: $(t, t^2)$

$m = y'(t)$

Tangent goes through $(1, -2)$

So slope $m = \frac{\text{slope between two points}}{1 - t}$

$$m = \frac{-2 - t^2}{1 - t}$$
Solution \[ y = x^2 \]
\[ y' = 2x \]

at \((t, t^2)\), \(y' = 2t\)

slope b/w \((t, t^2)\) and \((1, -2)\):

\[ m = \frac{-2 - t^2}{1 - t} = 2t \]

Solve for \(t\)

\[(1-t) \frac{-2 - t^2}{1-t} = 2t (1-t)\]

\[-2 - t^2 = 2t - 2t^2\]

\[t^2 - 2t - 2 = 0\]

Quadratic formula

\[ t = \frac{2 \pm \sqrt{4 - 4 \cdot 1 \cdot (-2)}}{2} \]

\[ t = \frac{2 \pm \sqrt{4 + 8}}{2} \]

\[ t = \frac{2 \pm \sqrt{12}}{2} \]

\[ t = 2 \pm \sqrt{3} \]
\[
t = \frac{2 \pm \sqrt{12}}{2} = \frac{2 \pm 2\sqrt{3}}{2} = 1 \pm \sqrt{3}
\]

\[
t_1 = 1 + \sqrt{3} \quad t_2 = 1 - \sqrt{3}
\]

\[
m_1 = 2(1 + \sqrt{3}) \quad m_2 = 2(1 - \sqrt{3})
\]

1st line:
\[
y - (1 + \sqrt{3})^2 = 2(1 + \sqrt{3})(x - (1 + \sqrt{3}))
\]

2nd line:
\[
y - (1 - \sqrt{3})^2 = 2(1 - \sqrt{3})(x - (1 - \sqrt{3}))
\]
Today: learn what functions can be differentiated.

☆ star review of exponential functions

Definition of derivative function

\[ f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \]

\( f'(x) = \) slope of tangent at \((x, f(x))\)

\( f'(x) = \) rate of change of \(f\) at \(x\)

Terminology: We say \(f(x)\) is differentiable at \(x = a\)

if \(f'(a)\) exists.

in other words, \(\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}\) exist.
What functions are differentiable?

\[ f(x) = \sqrt{x^2 - 1} \]

corner at \( x = 1 \)

slope of tangent = \( \lim_{h \to 0} \) (slope of secant lines)

simplest function with a corner: \( y = |x| = \begin{cases} x & x > 0 \\ 0 & x = 0 \\ -x & x < 0 \end{cases} \)

\[ y' = ? = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases} \]
\[ f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{f(h) - 0}{h} = \lim_{h \to 0^+} \frac{f(h)}{h} \]

\[ \lim_{h \to 0^+} \frac{f(h)}{h} \neq \lim_{h \to 0^-} \frac{f(h)}{h} \]

\[ \lim_{h \to 0^+} \frac{h}{h} = \lim_{h \to 0^+} 1 = 1 \]

\[ \lim_{h \to 0^-} \frac{-h}{h} = \lim_{h \to 0^-} -1 = -1 \]

\[ \lim_{h \to 0} \frac{f(h)}{h} = f'(0) = \text{DNE!} \]
so $|x|$ is \textbf{NOT} differentiable at $x=0$ where there is a corner.

- discontinuity at $x=1$
- the definition
  \[ f'(a) = \text{slope of tangent} \]
  \[ = \lim \text{(slope of secants)} \]
  fails.

If $f$ is \textbf{NOT} continuous at $x=a$,

$f$ is \textbf{NOT} differentiable at $x=a$.

$f$ must be continuous at $x=a$ to be differentiable.
Note: If $f$ is continuous at $x=a$, $f$ may or may not be differentiable at $x=a$.

Example: $y = f(x)$ is continuous at $x=0$, but it is **not** differentiable at $x=0$.

Theorem: If $f$ is differentiable at $x=a$, then $f$ is also continuous at $x=a$.

3. Vertical tangent at $x=0$:

$$y = \sqrt{x}$$

Inverse of $x^2, x>0$:

$$y' = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} \text{ is undefined at } x=0$$
in fact \( f'(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} \)

\[
= \lim_{h \to 0^+} \frac{f(h) - 0}{h} = \lim_{h \to 0^+} \frac{\sqrt{h}}{h} = \frac{1}{\sqrt{h}} \cdot \sqrt{h} = +\infty
\]

slope of tangent is \( \infty \)

\[
tangent \ line \ at \ x = 0 \ is \ vertical.
\]

3 cases:

- \( f \) is not differentiable at \( x = a \) if
- there is a corner at \( x = a \)
- there is a vertical tangent at \( x = a \)
- \( f \) is not continuous at \( x = a \).