Last time : estimated instantaneous velocity of an object.

Let $p(t)$ be the position of a moving object at time t

Consider change in position

$$\Delta p = p(t_f) - p(t_i)$$

for a change in time $\Delta t = t_f - t_i$

we define the <u>average velocity</u> of object over interval Δt as

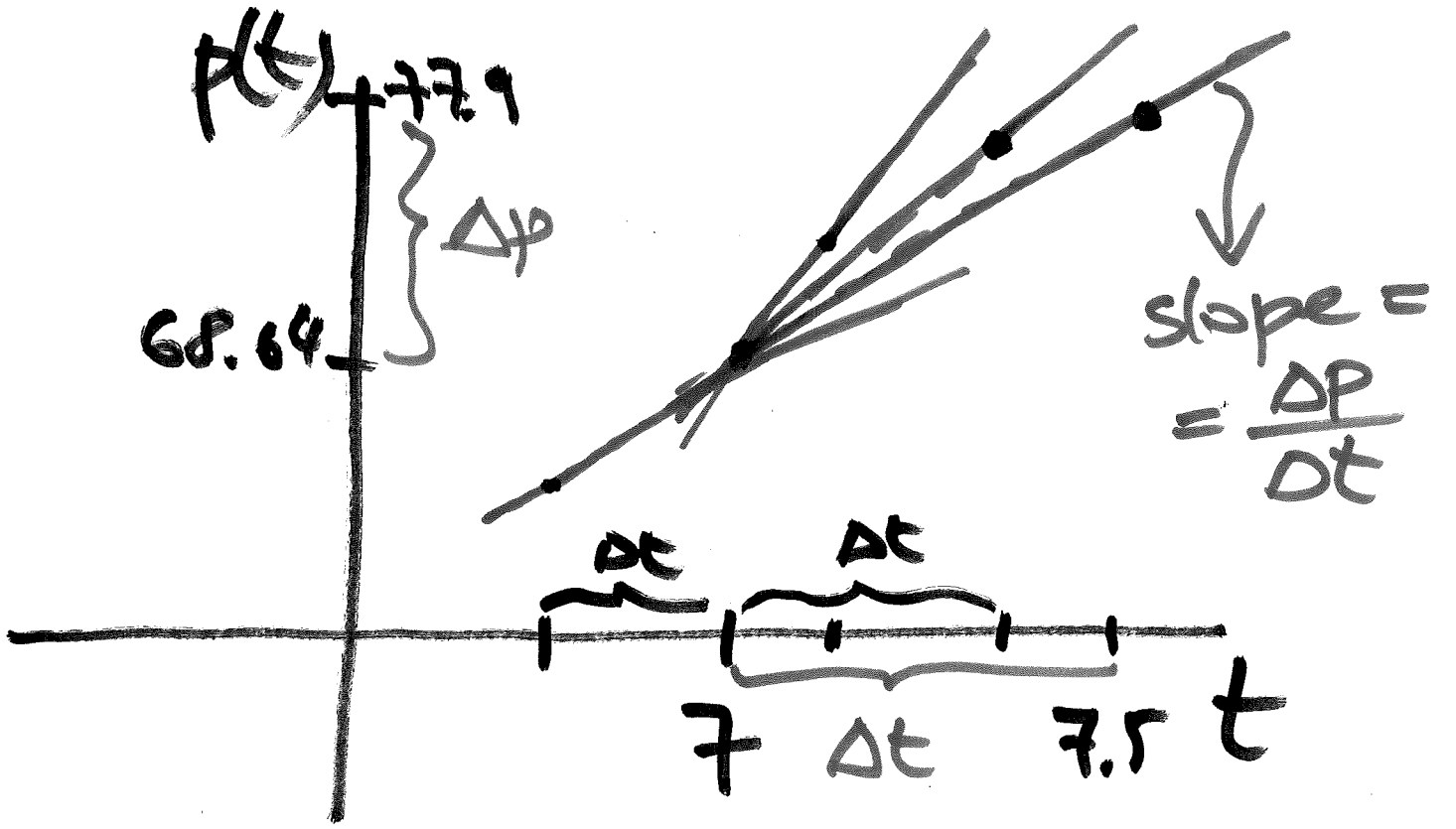
$$v_{avg} = \frac{\Delta p}{\Delta t} = \frac{p(t_f) - p(t_i)}{t_f - t_i}$$

by taking smaller and smaller Δt (* i.e. by making Δt approach zero) we found that the quantity $\frac{\Delta r}{\Delta t}$ approaches some number.

We <u>estimated</u> the instantaneous velocity to be that number.

Geometrically, this process corresponds to the following:

(computing slopes of secant lines)



Notation: a line that goes through two points on a curve is called a secant line

Mathematically, when we take smaller and smaller Δt (i.e. when Δt approaches zero), and we look at the value that the quantity $\frac{\Delta p}{\Delta t}$ approaches we are computing a "limit"

$$\lim_{\Delta t \to 0} \left(\frac{\Delta p}{\Delta t} \right) = v_{inst.}$$

reads "the limit of $\frac{\Delta p}{\Delta t}$ as Δt approaches zero * is v"

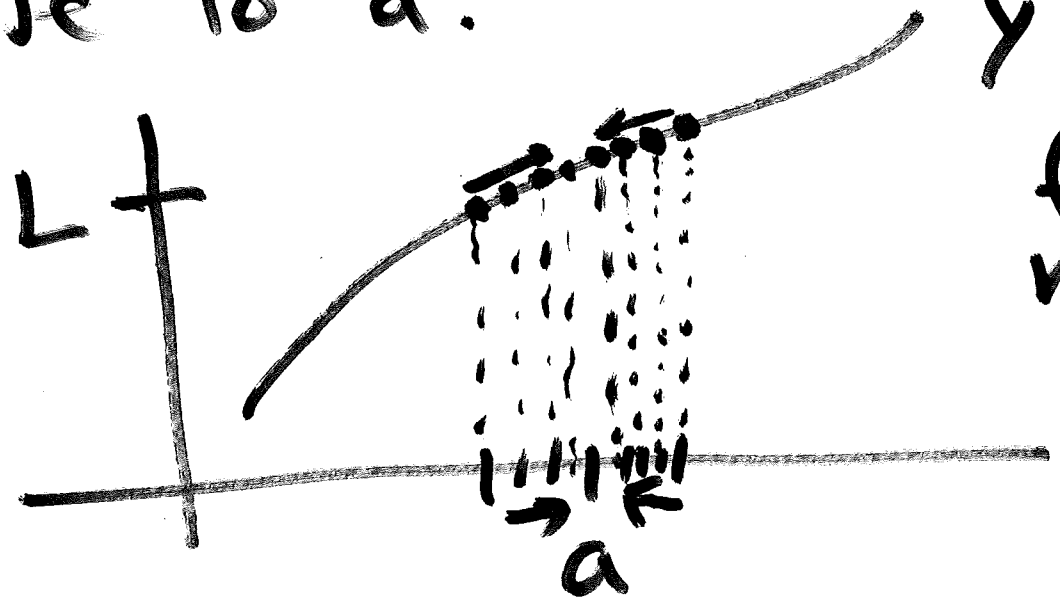
This week we'll learn how to compute limits.

Limit of a function

Consider a function $f(x)$,

we say $\lim_{x \to a} f(x) = L$

if we can make $f(x)$ be as close as we like to L by taking x-values sufficiently close to a.

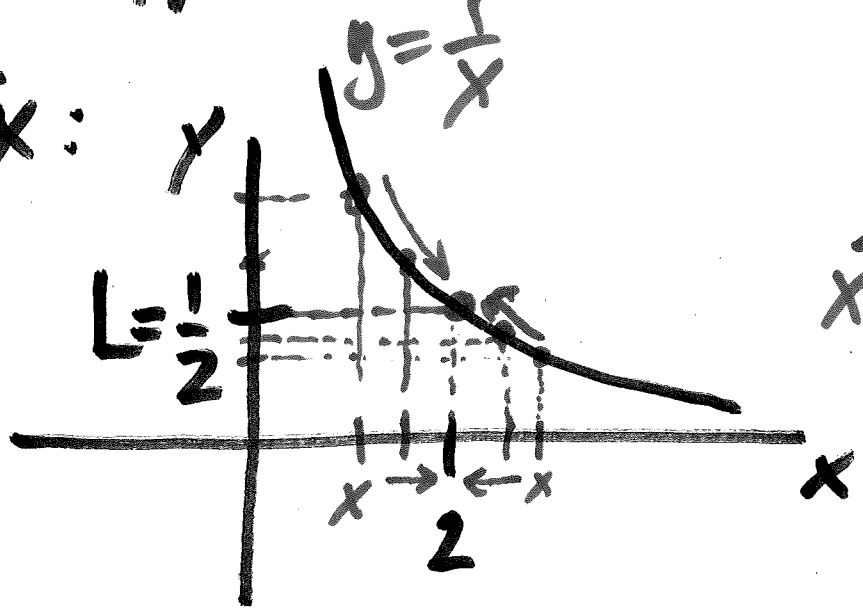


Today: basic ideas about the limit of a function

Notation: $\lim_{x \to a} f(x) = L$

reads "the limit of $f(x)$ as x approaches a is L"

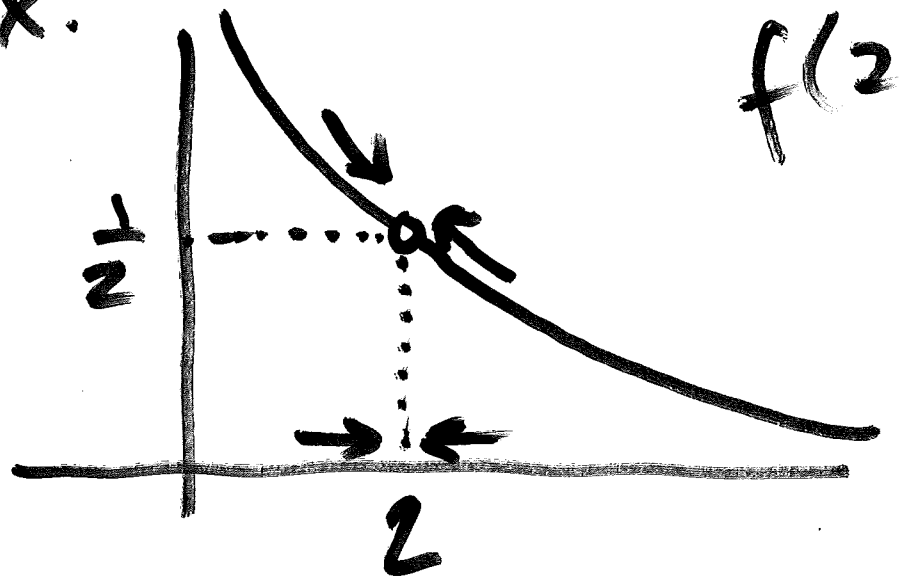
Ex:



$$\lim_{x \to 2} \frac{1}{x} = \frac{1}{2}$$

Remark 1: the exact value $f(a)$ is irrelevant to evaluate $\lim_{x \to a} f(x)$

Ex:

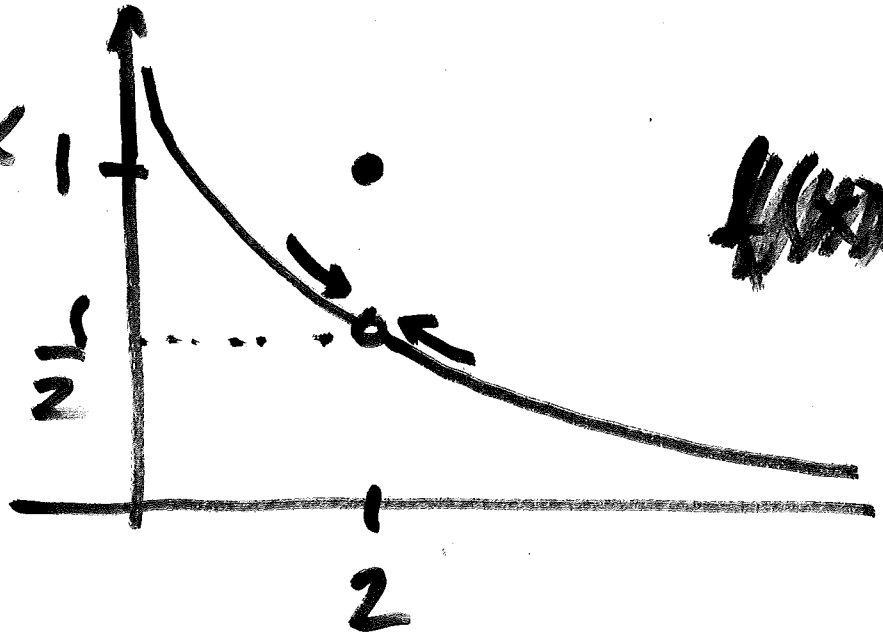


$f(2)$ is undefine

$\lim_{x \to 2} f(x) = \frac{1}{2}$

even though $f(2)$ DNE

Ex



$f(2) = 1$

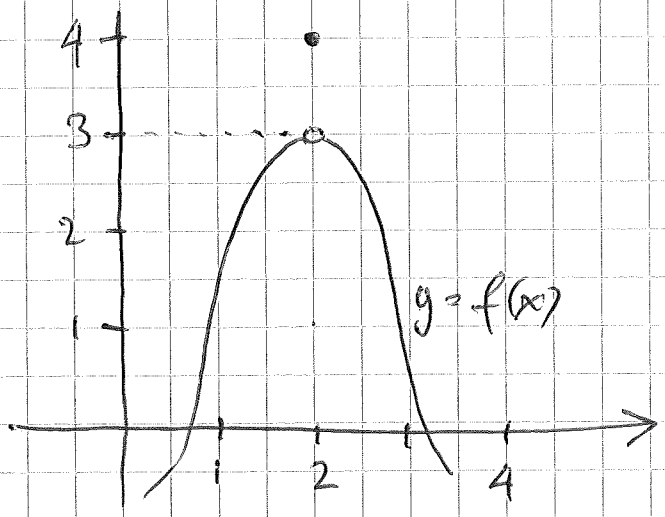
$\lim_{x \to 2} f(x) = \frac{1}{2}$

How to evaluate limits algebraically.

Ex: $\lim_{x \to 3} x^2 = 9$

x	x^2
2.99	$(2.99)^2 = 8.9401$
3.01	$(3.01)^2 = 9.0601$
2.999	$(2.999)^2 = 8.99\ldots$
3.001	$(3.001)^2 = 9.006$

Rule of thumb: to evaluate the limit of $f(x)$ as $x \to a$ simply compute $f(a)$, i.e. plug a into formula for f.



$\lim_{x \to 2} f(x) = ?$

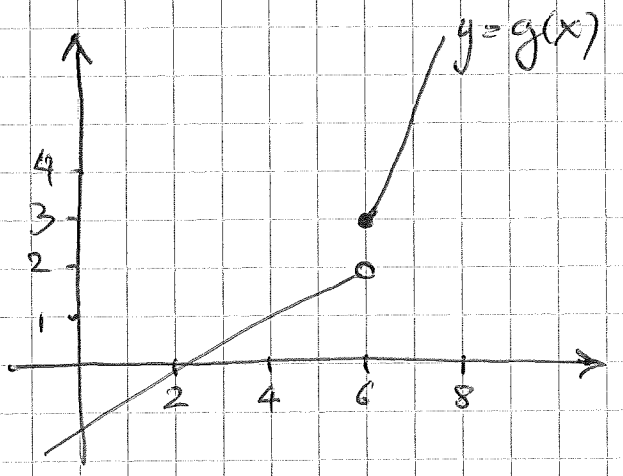
A) 0

B) 2

C) 3

D) 4

E) does not exist



$\lim_{x \to 4} g(x) = ?$

A) 0

B) 1

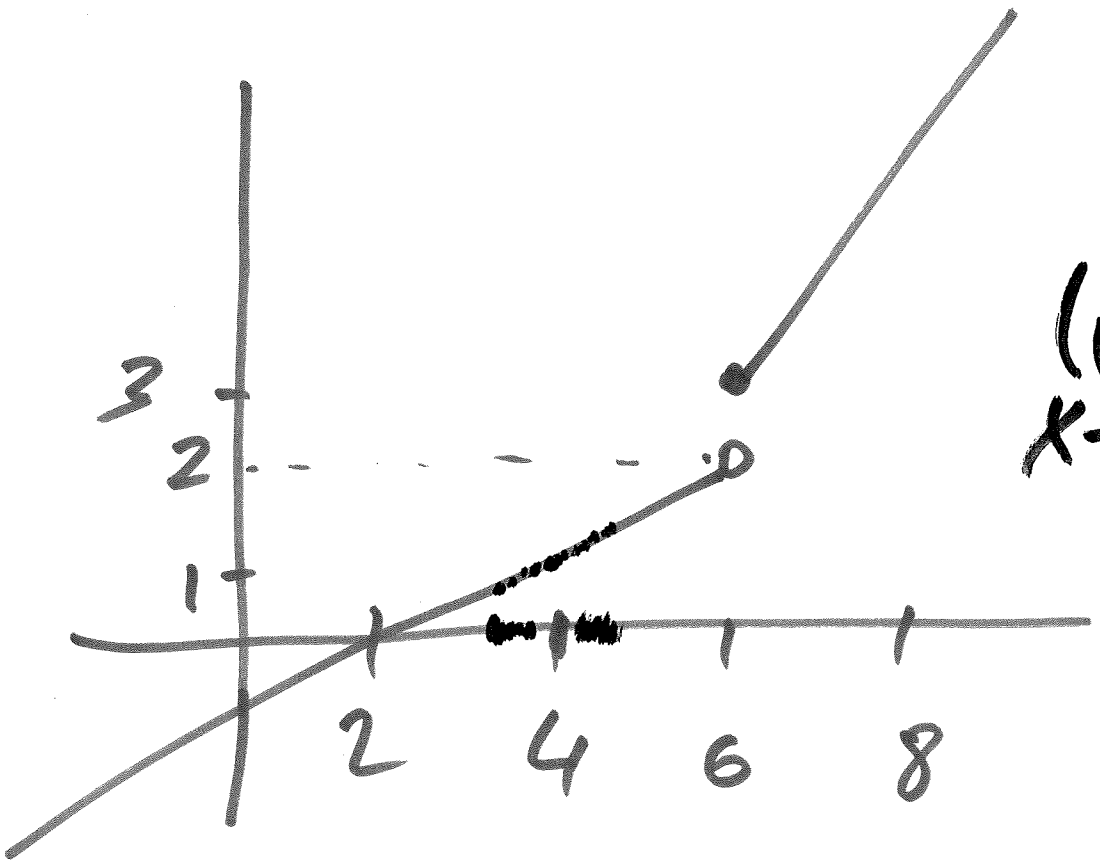
C) 2

D) 3

E) does not exist

$\lim_{x \to 6} g(x) = ?$

(same options as above)



$\lim_{x \to 4} f(x) = 1$

$\lim_{x \to 6} f(x)$ Does not exist (DNE)

because we cannot make $f(x)$ be as close as we like to a single number by taking x-values sufficiently close to 6.

$$\lim_{x \to 1} \frac{x^2+x-2}{x^2+5x+6} = ?$$

A) 0 ← $f(1) = \frac{1+1-2}{1+5+6} = \frac{0}{12}$

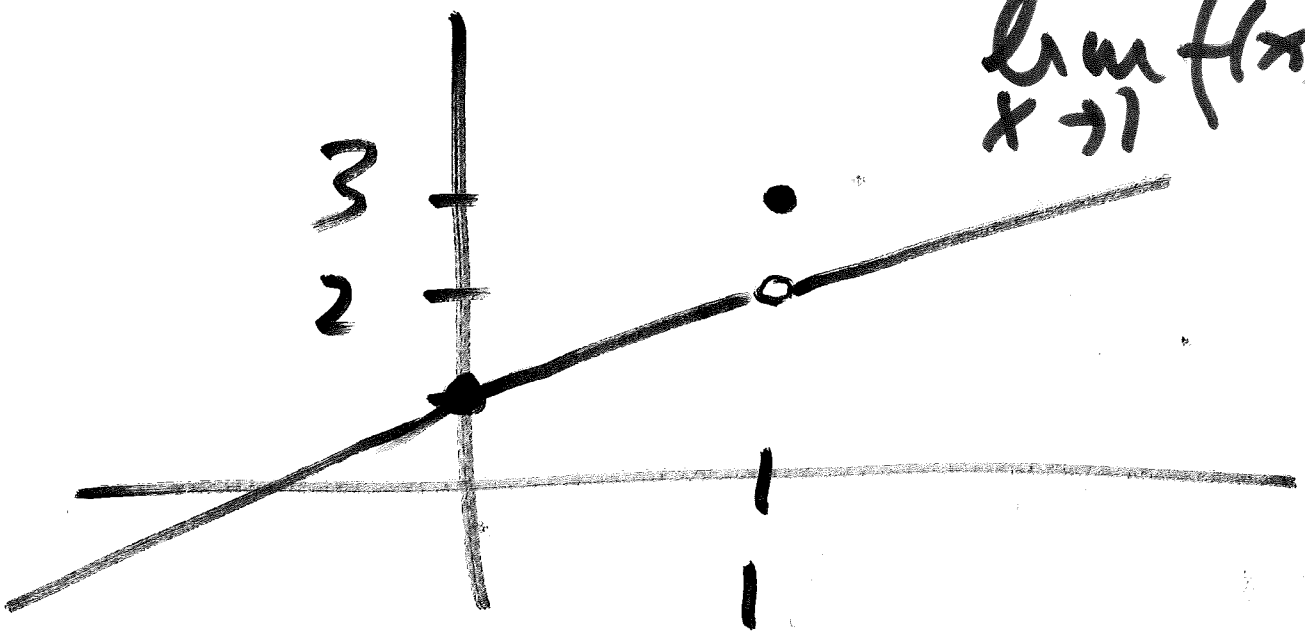
B) 1

C) 12

D) DNE

$$f(x) = \begin{cases} \frac{(x-1)(x+1)}{x-1} & x \neq 1 \\ 3 & x = 1 \end{cases}$$

$$f(x) = \begin{cases} x+1 & x \neq 1 \\ 3 & x = 1 \end{cases}$$



$\lim_{x \to 1} f(x) = 2$

$\lim_{x \to -2} \sqrt{x^2 - 3} = ?$

A) -1

B) 1 ←

C) DNE

$\sqrt{(-2)^2 - 3} = \sqrt{1} = 1$

$$\lim_{x \to -4} \frac{|x|+1}{\sqrt{x^2-2}} = \frac{|-4|+1}{\sqrt{(-4)^2-2}} =$$

$$= \frac{4+1}{\sqrt{16-2}} = \frac{5}{\sqrt{14}}$$

$$\lim_{x \to 1} f(x) = ?$$

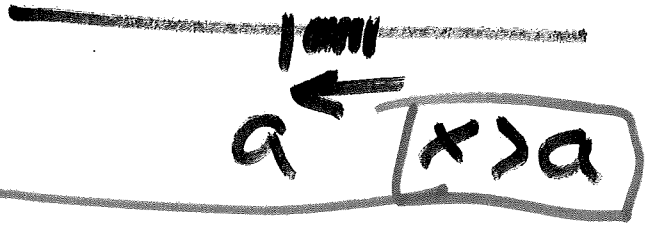
$$\text{if } f(x) = \begin{cases} x+1 & x \geq 1 \\ x^2 & x < 1 \end{cases}$$

$f(1) = 2$

here we take the limit from the right and the limit from the left

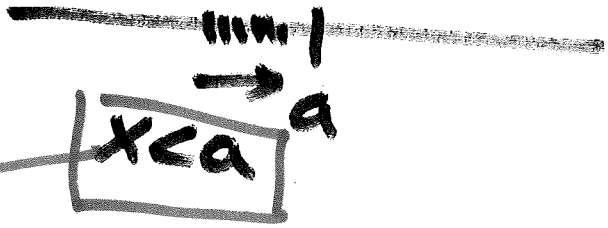
One-sided limits.

$\lim_{x \to a^+} f(x)$



right limit

$\lim_{x \to a^-} f(x)$



left-limit

So:

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} x+1 = 2$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} x^2 = 1$$

in this case we say

$\lim_{x \to 1} f(x)$ DNE

because $\lim_{x \to 1^+} f(x) \neq \lim_{x \to 1^-} f(x)$

Last time:

* discussed what $\lim_{x \to a} f(x) = L$ means

* rule of thumb to evaluate $\lim_{x \to a} f(x)$:

  1) if $f(a)$ is defined, plug a into formula and compute

  2) if $f(x)$ is piecewise and a is boundary value, take left and right limits and determine whether limit exists (left and right limits must match, otherwise limit DNE).

Evaluate $\lim_{x \to 1/2} \frac{4x^2-1}{3+2x}$

A) 0 ←

B) 4

C) DNE

D) 1/4

$$\frac{4\cdot(\frac{1}{2})^2-1}{3+2(\frac{1}{2})} = \frac{4\cdot\frac{1}{4}-1}{3+1}$$

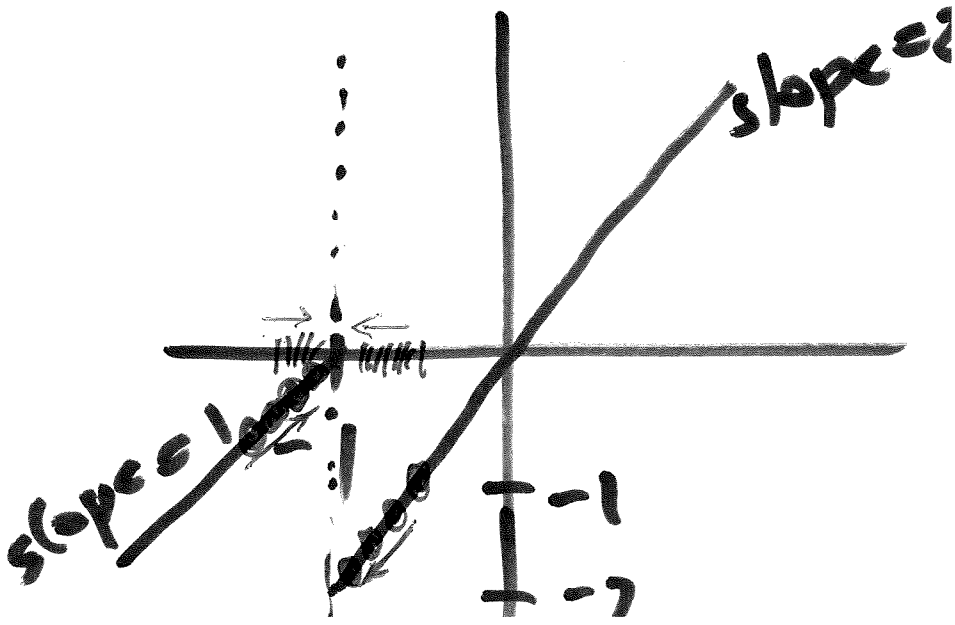
$\lim_{x \to -1} f(x)$ if $f(x) = \begin{cases} x+1 & x<-1 \\ 2x & x>-1 \end{cases}$

A) 0

B) -2

C) -1

D) DNE



Today : ~~evaluate~~

*) discuss limits that diverge to infinity

*) discuss indeterminate form "$\frac{0}{0}$"

Ex: $\lim_{x \to 1} \frac{(x-1)(x+2)}{x-1}$

$f(a)$ is undefined here.

let's try to simplify $f(x)$.

we factor and cancel common factors.

$$\lim_{x \to 1} \frac{\cancel{(x-1)}(x+2)}{\cancel{x-1}} = 3$$

In general, given $\lim_{x \to a} f(x)$ if $f(a)$ is undefined:

* you may end up with "$\frac{0}{0}$" then factor $f(x)$ and try to simplify.

ex: $\lim_{x \to 2} \frac{x-2}{x^2-3x+2}$

$f(2) = \frac{2-2}{4-6+2} = \frac{0}{0}$??

Let's factor and simplify

$\lim_{x \to 2} \frac{(x-2) \cdot 1}{(x-1)(x-2)} = \lim_{x \to 2} \frac{1}{x-1} = 1$

Why are we allowed to cancel the common factor?

it's because ✱ when we evaluate the limit $\lim_{x \to a} f(x)$,

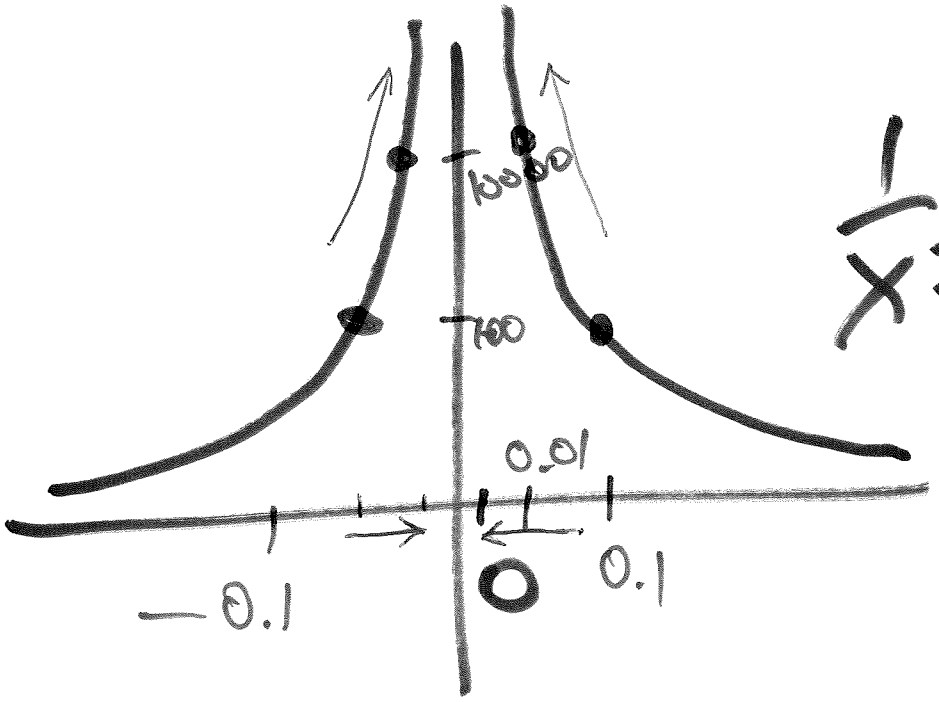
$x \neq a$ so $x - a \neq 0$

$\lim_{x \to 0} \frac{1}{x^2}$

- $f(0)$ is undef.
- $f(x)$ can't be simplified further

So let's go back to basic principles about limits.

x	0.1	0.01	-0.01
$1/x^2$	$\frac{1}{(\frac{1}{10})^2} = 100$	$\frac{1}{(\frac{1}{100})^2} = 10\,000$	10 000



$\frac{1}{x^2}$

as x gets closer to 0, $f(x)$ grows larger and larger in the positive direction.

we say $\lim_{x \to 0} f(x)$ diverges to ∞

↖ infinity (positive)

$\lim_{x \to 0} \frac{1}{x^2}$ "$=$" ∞

↑ infinity is not a

~~Ingeneral~~ In general, given $\lim_{x \to a} f(x)$, if $f(a)$ is undefined

*) check if you have "$\frac{0}{0}$"

*) check if limit diverges to ∞

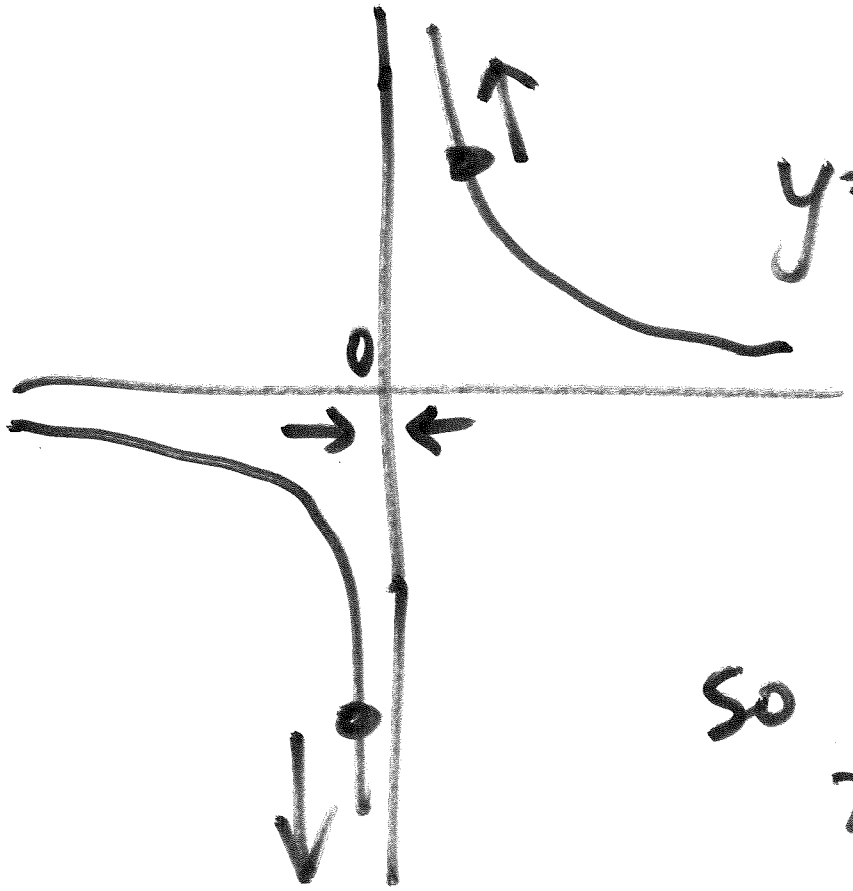
---

ex: $\lim_{x \to 0} \frac{1}{x}$

$f(0)$ is undefined, can't simplify

so let's build a table of values

x	0.1	0.01	0.001	-0.1	-0.001
$1/x$	10	100	1000	-10	-1000



so $\lim_{x \to 0} \frac{1}{x}$ DNE

but $\lim_{x \to 0^+} \frac{1}{x}$ diverges to ∞

$\lim_{x \to 0^-} \frac{1}{x}$ diverges to $-\infty$