Today: look at the relationship between instantaneous velocity (or instantaneous rate of change of a function) and tangent lines. Introduce the idea of limit of a function.

Last time:
we observed that computing average velocity over a time interval \([t_1, t_2]\) corresponds to computing the slope of line that goes through the points \((t_1, f(t_1))\) and \((t_2, f(t_2))\). We call this line the secant line.

Suppose we can fit a curve through the data points, \(y = f(t)\).
consider a time interval \([t_0, t_1]\)

slope between \(P\) and \(Q\)

\[
m = \frac{f(t_1) - f(t_0)}{t_1 - t_0}
\]

this equals average velocity over the interval \([t_0, t_1]\)

As we shorten \(\Delta t = t_1 - t_0\),
the point \(Q\) gets closer to \(P\).

the slope \(m = \frac{f(t_1) - f(t_0)}{t_1 - t_0}\) of secant line
becomes the slope of tangent line.
In other words,
as $t$ gets closer to $t_0$,
$Q$ gets closer to $P$,
the secant line through $P$ and $Q$
becomes the line that touches
the curve at $P$ which we call
tangent line. $P$ is the point of
tangency.

So the process of estimating the
instantaneous velocity at $(t, f(t_0))$
corresponds to estimating
the slope of the tangent line
to the curve $y = f(t)$ at $(t_0, f(t_0))$.

We expect the velocity at $t_0$ to be
the slope of the tangent line
at $(t_0, f(t_0))$. 
By shortening $\Delta t$ to be close to 0, we compute the slope of secant lines that get close to tangent line.

In the limiting case, we get exactly the slope of the tangent line.

$$\lim_{t_1 \to t_0} \left( \frac{f(t_1) - f(t_0)}{t_1 - t_0} \right) = \text{slope of tangent line at } (t_0, f(t_0))$$

equivalently

$$\lim_{t_1 \to t_0} \left( \frac{f(t_1) - f(t_0)}{t_1 - t_0} \right) = \text{velocity at } t = t_0$$
Notation:

\[ t_1 = t_0 + h \]

where \( h \) is a small number, either positive or negative.

Then, the slope of the secant line through \( P \) and \( Q \) is:

\[
m = \frac{f(t_0 + h) - f(t_0)}{t_0 + h - t_0} = \frac{f(t_0 + h) - f(t)}{h}
\]

Then:

\[
\lim_{h \to 0} \left( \frac{f(t_0 + h) - f(t_0)}{h} \right) = \text{velocity at } t_0 = \text{slope of tangent line at } (t_0, f(t_0))
\]
**Limit of a function**

\[ \lim_{x \to a} f(x) \]

reads "the limit of \( f \) as \( x \) approaches \( a \)."

This describes the behaviour of a function \( f \) when the independent variable \( x \) gets closer and closer to some number "\( a \)."

\[ y = f(x) \]

\[ \lim_{x \to 2} f(x) = ? \]

as \( x \) gets closer and closer to 2, \( f(x) \) gets closer to \( \frac{1}{2} \).

So we say

\[ \lim_{x \to 2} f(x) = \frac{1}{2} \]
Today:
* describe the meaning of \( \lim_{x \to a} f(x) = L \)
* evaluate limits at a point using substitution property or algebraic manipulations

Last time: Introduced the concept of limit of a function.

\( \lim_{x \to a} f(x) = L \) means that we can make \( f \) be as close as we like to \( L \) by taking values of \( x \) close to "\( a \)"

\[
\begin{array}{c|c}
X & \frac{1}{X} \\
1.99 & 0.5025... \\
1.999 & 0.50025... \\
2.01 & 0.4975... \\
2.001 & 0.4997... \\
\end{array}
\]

\( \lim_{x \to 2} \frac{1}{x} = \frac{1}{2} \)
**Observation #1:**

The exact value $f(a)$ is irrelevant when computing $\lim_{x \to a} f(x)$.

- $f(a) = 2$, $\lim_{x \to 2} f(x) = 3$
- $f(a) = 4$, $\lim_{x \to 2} f(x) = 3$
- $f(a)$ is undefined, $\lim_{x \to 2} f(x) = 3$
\[
\lim_{{x \to 4}} g(x) = 1
\]
\[
g(6) = 3
\]

\[
\lim_{{x \to 6}} g(x) = \text{does not exist (DNE)}
\]

because we cannot make \( f(x) \) get closer to a specific number by taking \( x \)-values close to 6 (on both sides of 6).

Observation #2: if we cannot make \( f(x) \) approach a specific number \( L \) by taking values of \( x \) close to \( a \), we say that \( \lim_{{x \to a}} f(x) = \text{DNE} \).
How to evaluate limits

For most functions, \( \lim_{x \to a} f(x) \) can be evaluated directly by plugging "a" into \( f \).

Ex: \( \lim_{x \to 8} x^2 = 64 \)

\( \lim_{x \to 3} \sqrt{x^2+1} = \sqrt{10} \)

This is called "Substitution property".

Substitution works whenever \( f(x) \) is defined at \( x = a \)

\( \lim_{x \to 3} \frac{1}{x-1} = \frac{1}{3-1} = \frac{1}{2} \)

\( \lim_{x \to 0} \frac{|x|}{\sqrt{1-x^2}} = \frac{0}{\sqrt{1}} = 0 \)
What if \( f(a) \) is undefined?

**Case 1**

*E.g.* \( \lim_{x \to 2} \frac{x^2 - 4}{x - 2} \)

\[
\frac{x^2 - 4}{x - 2} \text{ is undefined at } x = 2
\]

Strategy: manipulate the function and simplify.

*E.g.* \[
\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2
\]

allowed because \( x \neq 2 \)

can rewrite the limit as:

\[
\lim_{x \to 2} \frac{x^2 - 4}{x - 2} = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \to 2} (x + 2)
\]

\( = 4 \)
Examples

1) \( \lim_{x \to -2} \frac{x^2 + 2x}{x^2 - 4} = \lim_{x \to -2} \frac{x(x+2)}{(x-2)(x+2)} = \lim_{x \to -2} \frac{x}{x-2} = \frac{-2}{-2-2} = \frac{1}{2} \)

   because \( x \neq -2 \)

2) \( \lim_{t \to 0} \frac{3t + 4t^2}{t - t^2} = \lim_{t \to 0} \frac{x(3+4t)}{t(1-t)} = \lim_{t \to 0} \frac{3+4t}{1-t} = \frac{3+0}{1-0} = 3 \)

   because \( t \neq 0 \)

3) \( \lim_{x \to 2} \frac{x^4 - 16}{x^3 - 8} = \)

   Recall: \( a^2 - b^2 = (a+b)(a-b) \) \(<\) memorize this.

   \( a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 - b^2)(a^2 + b^2) \)

   \( = (a+b)(a-b)(a^2 + b^2) \)

   do not memorize this
\[
\lim_{x \to 4} \frac{x^4 - 16}{x^3 - 8} = \lim_{x \to 4} \frac{(x^2 - 4)(x^2 + 4)}{x^3 - 8} \\
= \lim_{x \to 4} \frac{(x - 2)(x + 2)(x^2 + 4)}{x^3 - 8}
\]

Recall

\[a^3 - b^3 = (a - b)(a^2 + ab + b^2)\]

back to limit

\[
\lim_{x \to 2} \frac{x^4 - 16}{x^3 - 8} = \lim_{x \to 2} \frac{(x - 2)(x + 2)(x^2 + 4)}{(x - 2)(x^2 + 2x + 4)}
\]

\[
= \frac{(2 + 2)(2^2 + 4)}{2^2 + 2 \cdot 2 + 4} = \frac{8}{12} = \frac{2}{3}
\]
From Q1

\[ f(x) = \begin{cases} 
-x^2 + x + 2 & x < 1 \\
-x + 6 & x > 1 
\end{cases} \]

\( f(1) \) is undefined

Domain: all \( x \neq 1 \)

\[-x^2 + x + 2 = 0\]
\[-(x^2 - x - 2) = 0\]
\[-(x - 2)(x + 1) = 0\]
\( x = 2 \)
\( x = -1 \)

<table>
<thead>
<tr>
<th>( x )</th>
<th>( f(x) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-1 + 1 + 2 = 2</td>
</tr>
<tr>
<td>0</td>
<td>-2 + 6 = 4</td>
</tr>
<tr>
<td>1</td>
<td>-1 + 6 = 5</td>
</tr>
</tbody>
</table>

\( -x + 6 = 0 \)
\( x = 6 \)

don't
for piecewise functions:
1) indicate and label intercepts
2) complete graph for "all" x-values
3) make sure your graph is the graph of a function:
4) make a table of values.
Today: 1) Interpret the meaning of \( \lim_{x \to a} f(x) = 0 \)
2) Compute one-sided limits

Last time: How to evaluate limits.

Substitution Property:
if \( f(a) \) is defined, in most cases \( \lim_{x \to a} f(x) = f(a) \)

This is true if \( f(x) \) is polynomial function
rational function
for some piecewise functions

What if \( f(a) \) is undefined?

Case 1: use algebraic manipulation
to simplify the function

\[
\lim_{x \to \sqrt{2}} \frac{x - 2}{x - \sqrt{2}} = \lim_{x \to \sqrt{2}} \frac{(x - \sqrt{2})(x + \sqrt{2})}{x - \sqrt{2}} = 2\sqrt{2}
\]
\[
\lim_{x \to 0} \frac{1}{x^2} = ?
\]

```
<table>
<thead>
<tr>
<th>x</th>
<th>(1/x^2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.1</td>
<td>100</td>
</tr>
<tr>
<td>0.01</td>
<td>10000</td>
</tr>
<tr>
<td>0.001</td>
<td>1000000</td>
</tr>
<tr>
<td>-0.1</td>
<td>100</td>
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<td>10000</td>
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<tr>
<td>-0.001</td>
<td>1000000</td>
</tr>
</tbody>
</table>
```

As \(x\) gets close to 0, \(f(x)\) grows

\[
\lim_{x \to 0} x + \sqrt{2} = \sqrt{2} + \sqrt{2} = 2\sqrt{2}
\]
we cannot make \( f(x) \) get close to a specific number. So we say \( \lim_{x \to 0} \frac{1}{x^2} \) does not exist.

More specifically, \( f(x) \) grows very large as \( x \) gets closer and closer to 0, so we say \( \lim_{x \to 0} \frac{1}{x^2} = \infty \) (not a number).

Similarly, \( \lim_{x \to 0} \frac{-1}{x^2} = -\infty \) (a very large negative quantity).
In general, when \[ \lim_{x \to a} f(x) = \infty \text{ or } -\infty \]
we say that \( f(x) \) has a vertical asymptote at \( x = a \).

\[ \lim_{x \to a} f(x) = \infty \]

\[ \downarrow \text{ vertical line } x = a. \]

Ex: \[ \lim_{x \to 2} \frac{x^2 - 4}{(x-2)^3} = \lim_{x \to 2} \frac{(x-2)(x+2)}{(x-2)^3} = \]

\[ \lim_{x \to 2} \frac{x+2}{(x-2)^2} = \infty \]

\[ \begin{array}{c|c}
  x & \frac{(x+2)}{(x-2)^2} \\
\hline
  2.01 & 4.01 / (0.01)^2 = 401000 \text{ grows large} \\
  2.001 & 4.001 / (0.001)^2 = 4001000 \text{ grows large} \\
  1.99 & 3.99 / (-0.01)^2 = 39900 \text{ grows large} \\
  1.999 & 3.999 / (-0.001)^2 = 399900 \text{ grows large} \\
\end{array} \]
One-sided limits

$$\lim_{x \to a^+} f(x)$$
right-end limit

$$\lim_{x \to a^-} f(x)$$
left-end limit.

$$\lim_{x \to 0^+} \frac{1}{x} = +\infty$$

$$\lim_{x \to 0^-} \frac{1}{x} = -\infty$$

$$\lim_{x \to 1} f(x) = \text{DNE}$$
this is not $$\infty$$
But \( \lim_{x \to 0} \frac{1}{x} = \text{DNE} \)