Mon March 19

Today: solving optimization problems with constraints

Problem: Find two numbers whose difference is 100 and whose product is the smallest possible.

1. identify quantity that needs to be minimize => product of two numbers
2. sketch => not useful
3. variables : x an y are numbers
4. write a formula for product

$P = x \cdot y$ we have two variables x and y are they independent? No!

$x - y = 100$ ← constraint

$x = 100 + y$

(5) rewrite formula found in (4) as a function of <u>one</u> variable using the constraint

$$P = x \cdot y = (100+y) \cdot y$$

(6) $P(y) = (100+y)y$ domain: all y

(7) use calculus to find global min.

- find critical #s.

$$P'(y) = 100 + 2y$$

$$P'(y) = 100 + 2y = 0$$

$$y = -50$$

$$P(-50) = (100-50)(-50) = -250$$

- find asymptotic behaviour

$$\lim_{x \to \infty} P(y) = \lim_{x \to \infty} 100y + y^2 = \infty$$

$$\lim_{x \to -\infty} P(y) = \lim_{x \to -\infty} 100y + y^2 = \infty$$

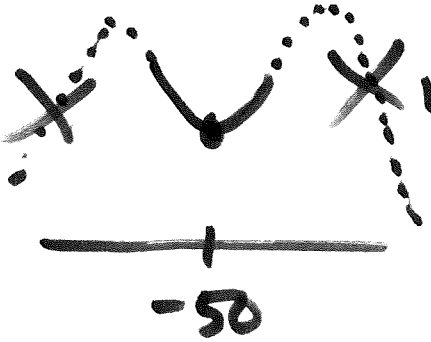
- compare $P(-50) = -2500 < \infty$

global min!

two numbers are $y = -50$

$x = 100 - 50 = 50$

Remark: $P(y)$ has only one critical number. Is it enough to prove that the critical number corresponds to a local minimum to claim it is also a global minimum? yes!



not possible because there no other critical numbers besides -50

So instead of comparing $P(-50)$ with asymptotic behaviour of $P(y)$ as $y \to \infty$ and $y \to -\infty$, we could just prove that $x = -50$ corresponds to a local minimum

$\Rightarrow$ apply 1st derivative test

⇒ apply 2nd derivative test
(either tests can be used to find local extrema)

Bottom line: If a function has only one critical number $x=c$, we can prove $f(c)$ is a global extremum by proving that $f(c)$ is a local extremum.

Worksheet: Optimization Problems

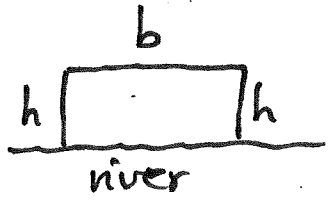
Problem 1

A farmer has 60 metres of fencing and wants to fence off a rectangular field that borders a straight river. No fence is needed along the river. What are the dimensions of the field that has the largest area?

1. **IDENTIFY WHAT QUANTITY NEEDS TO BE OPTIMIZED**

   The quantity that needs to be (*maximized*) *minimized* is area of field (complete in words)

2. **SKETCH** and 3. **LABEL VARIABLES**



The relevant variables are b, h, A

Don't forget to label the quantity that needs optimizing!

4. **WRITE DOWN A FUNCTION** REPRESENTING THE QUANTITY THAT NEEDS TO BE OPTIMIZED.

$A = b \cdot h$ (area)

How many variables does your function have? 2 (b and h)

If more than one variable, then we need to ***reduce the number of variables*** *using constraints.* → need one constraint

5. **IDENTIFY CONSTRAINTS.** In this problem the constraint is length of fencing material

$2h + b = 60 \longrightarrow b = 60 - 2h$

6. **REWRITE FUNCTION** IDENTIFIED IN 4.

$A = b \cdot h = (60-2h) \cdot h = 60h - 2h^2$

7. **IDENTIFY THE DOMAIN** OF THE FUNCTION

can h take any value? No! $h > 0$ also $b > 0$

↳ $60 - 2h > 0$

$h < 30$

domain $0 \leq h \leq 30$

endpoints are acceptable (math works out OK.)

8. **USE CALCULUS** to find global extremum.

$A' = 60 - 4h = 0$

$h = 15$

$A(15) = 60 \cdot 15 - 2 \cdot 15^2 = 2 \cdot 15 (30 - 15) = 2 \cdot 15 \cdot 15 > 0$ ← global maximum

$A(0) = 0$, $A(30) = 0$

Wed, March 21

Today: contd practice on optimization problems, especially problems involving more than one variable and with constraints

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Problem-Solving Strategy for Optimization Problems.

1. identify quantity that needs to be maximized / minimized.
2. sketch
3. label relevant variables
4. write a function that models the quantity that needs optimizing.
5. check how many variables are in the function written in step 4. if more than one variable, then identify constraints in order to relate variables.

(6) rewrite function identified in (4) as a function of one variable by substituting the other variables.

(7) identify domain of function

(8) use calculus to find absolute max/min.

Worksheet: Optimization Problems

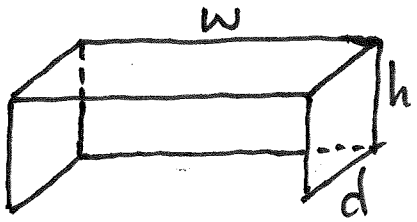
Problem 1

Shane is building a rain shelter to store 250 m^3 of lumber over the winter. The shelter is to have a back, two square sides and a top. The sides of the shelter will all be made out of plastic that costs \$5 per m^2, while the metal materials for the roof costs \$10 per m^2. What are the dimensions of the shelter that will cost Shane the least amount of money?

1. **IDENTIFY WHAT QUANTITY NEEDS TO BE OPTIMIZED**

The quantity that needs to be *maximized/minimized* is cost of construction (complete in words)

2. **SKETCH** and 3. **LABEL VARIABLES**



no front (no door)
no floor

cost to build sides: \$5/m^2
cost to build roof: \$10/m^2

The relevant variables are w, h, d, Cost

Don't forget to label the quantity that needs optimizing!

4. **WRITE DOWN A FUNCTION** REPRESENTING THE QUANTITY THAT NEEDS TO BE OPTIMIZED.

Cost

$C = 5 \cdot 2(d \cdot h) + 5(w \cdot h) + 10(w \cdot d)$

How many variables does your function have? 3 (w, d, h)

If more than one variable, then we need to ***reduce the number of variables*** *using constraints.*

⇒ need two constraints

5. **IDENTIFY CONSTRAINTS**. In this problem the constraints are:

① volume, $250 = w \cdot d \cdot h \rightarrow w \cdot h^2 = 250$

② square sides, $d = h$

↳ $w = \frac{250}{h^2}$ (easier) or $h = \sqrt{\frac{250}{w}}$?

6. **REWRITE FUNCTION** IDENTIFIED IN 4.

$C = 10(h^2) + 5\left(\frac{250}{h^2} \cdot h\right) + 10\left(\frac{250}{h^2} \cdot h\right) = 10h^2 + \frac{1250}{h} + \frac{2500}{h} = 10h^2 + \frac{3750}{h}$

7. **IDENTIFY THE DOMAIN** OF THE FUNCTION

$h > 0$ no other restrictions so domain: $(0, +\infty)$

$w > 0$

8. **USE CALCULUS** to find global extremum.

$C' = 20h - \frac{3750}{h^2} = 0$, C' is undefined at $h = 0$ ⇒ not in domain, reject

$20h^3 - 3750 = 0$

$h^3 = \frac{3750}{20} = \frac{375}{2}$

$h = \sqrt[3]{\frac{375}{2}}$

asymptotic behaviour:

$\lim_{h \to 0^+} 10h^2 + \frac{3750}{h} = +\infty$

$\lim_{h \to +\infty} 10h^2 + \frac{3750}{h} = +\infty$

⇒ $C\left(\sqrt[3]{\frac{375}{2}}\right)$ is

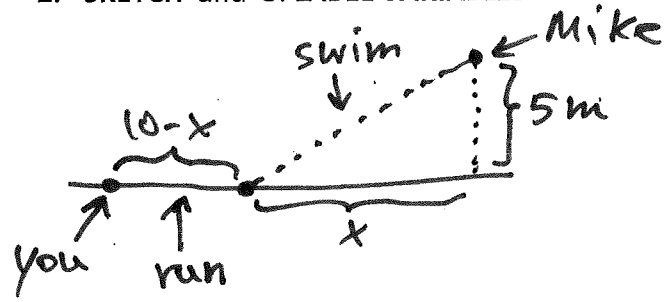
Problem 2

You are walking on the shore along a straight stretch of beach, when you see ~~a swimmer~~ Mike in distress. ~~The swimmer~~ Mike appears to be roughly 5 metres out to sea (on a straight line between the swimmer and the shore), and you are roughly 10 metres down the beach from the nearest point on shore to the swimmer. You can run at 5 metre per second along the beach, and you can swim at 3 metre per second. Given that you want to reach the swimmer as quickly as possible, how far down the beach do you run, and how far do you swim?

1. **IDENTIFY WHAT QUANTITY NEEDS TO BE OPTIMIZED**

The quantity that needs to be *maximized/(minimized)* is time to get to Mike (complete in words)

2. **SKETCH** and 3. **LABEL VARIABLES**



$v_{run} = 5$ m/s
$v_{swim} = 3$ m/s
} both constant

travel time = distance / speed

i. The relevant variables are
x, t

b. *Don't forget to label the quantity that needs optimizing!*

time spent running = $\frac{10-x}{v_{run}}$

time spent swimming = $\frac{\sqrt{5^2+x^2}}{v_{swim}}$

3. **WRITE DOWN A FUNCTION** REPRESENTING THE QUANTITY THAT NEEDS TO BE OPTIMIZED.

$$t = \frac{10-x}{5} + \frac{\sqrt{25+x^2}}{3}$$

1. How many ~~x~~ independent variables does your function have? 1
If more than one variable, then we need to ***reduce the number of variables*** *using constraints.*

4. **IDENTIFY CONSTRAINTS**. In this problem the constraint is none

(in practise, your running speed and swimming speed are your limiting factors, but here the speeds are known so they don't appear as constraints)

5. **REWRITE FUNCTION** IDENTIFIED IN 4.

$$t(x) = \frac{10-x}{5} + \frac{\sqrt{25+x^2}}{3}$$

6. **IDENTIFY THE DOMAIN** OF THE FUNCTION and 7. USE CALCULUS to find min.

Domain $0 \le x \le 10$

$$t'(x) = -\frac{1}{5} + \frac{1}{3}\cdot\frac{2x}{2\sqrt{x^2+25}} = -\frac{1}{5} + \frac{x}{3\sqrt{x^2+25}} = 0$$

$$\frac{x}{3\sqrt{x^2+25}} = \frac{1}{5}$$

$$\frac{x^2}{9(x^2+25)} = \frac{1}{25}$$

$$25x^2 = 9x^2 + 9\cdot 25$$

$x = \sqrt{9\cdot 25} = 15$

$x = -15$ also solution, but not in domain so reject.

Compare y-values

$$t(0) = \frac{10}{5} + \frac{\sqrt{25}}{3} = \frac{11}{3} > \frac{10}{3}$$

$$t(10) = 0 + \frac{\sqrt{25+100}}{3} = \frac{5\sqrt{5}}{3} > \frac{5\sqrt{4}}{3} = \frac{10}{3}$$

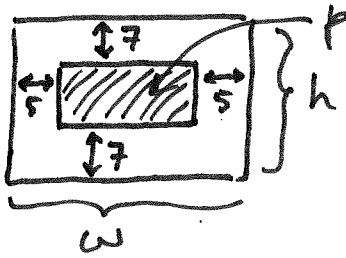
$t(15) = \ldots = 10/3$ ← global minimum

Problem 2

The top and bottom margins of a rectangular poster are each 7 cm and the side margins are each 5 cm. If the poster has to have an area of 1000 cm^2, what dimensions of the poster will give the largest printed area?

1. **IDENTIFY WHAT QUANTITY NEEDS TO BE OPTIMIZED**
   In this problem we need to *maximize*/*minimize* the printed area

2. **SKETCH** and 3. **LABEL VARIABLES.**



printed area

The relevant variables are $w, h, A_{printed}$

height of printed area $= h - 2\cdot 7 = h - 14$

width of printed area $= w - 2\cdot 5 = w - 10$

4. **WRITE DOWN A FUNCTION** REPRESENTING THE QUANTITY THAT NEEDS TO BE OPTIMIZED.

$A_{printed} = (h-14)(w-10)$

How many variables does your function have? 2

If more than one variable, then we need to ***reduce the number of variables*** *using constraints.* → need one constraint

5. **IDENTIFY CONSTRAINTS**: In this problem the constraint is total area of poster (complete). (1000 cm^2)

$w \cdot h = 1000 \rightarrow w = \frac{1000}{h}$ (now substitute into function in step 4.)

6. **REWRITE FUNCTION** IDENTIFIED IN 4.

$$A_{printed} = (h-14)\left(\frac{1000}{h} - 10\right) = 1000 - \frac{14000}{h} - 10h + 140$$
$$= 1140 - \frac{14000}{h} - 10h$$

7. **IDENTIFY THE DOMAIN** OF THE FUNCTION

$h > 0$ also $h - 14 \geq 0 \rightarrow h \geq 14$ (endpoints are o.k., math still works out ok)

$w > 0$ $\quad w - 10 \geq 0 \rightarrow w \geq 10 \rightarrow \frac{1000}{h} \geq 10$

$h \leq 100 \rightarrow$ domain $14 \leq h \leq 100$

8. **USE CALCULUS** to find global extremum.

$$A'_{printed} = \frac{14000}{h^2} - 10$$

$\frac{14000}{h^2} - 10 = 0$

$h^2 = 1400$

$h = \sqrt{1400} = 10\sqrt{14}$

$h = -\sqrt{1400}$ ← not in domain, reject.

$A'_{printed}$ is undefined at $h = 0$, not in domain so reject.

$A_{printed}(14) = 0 \cdot \left(\frac{1000}{14} - 10\right) = 0$

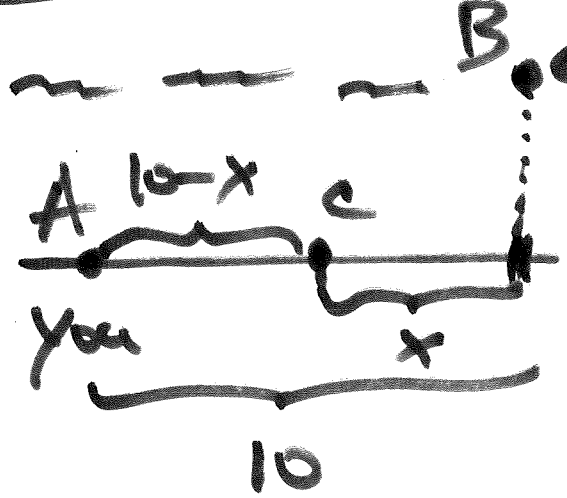
$A_{printed}(100) = (100-14)\left(\frac{1000}{100} - 10\right) = 0$

$A_{printed}(10\sqrt{14}) > 0$ ← global maximum

so answer is $h = 10\sqrt{14}$ and $w = \frac{100}{\sqrt{14}}$

Fri March 23

Today: solving optimization problems involving free parameters.



Minimize travel time to go from A to B via C.

$$t(x) = \frac{10-x}{5} + \frac{\sqrt{25+x^2}}{3}$$

$$= \frac{10}{5} - \frac{x}{5} + \frac{1}{3}\sqrt{25+x^2}$$

$$= 2 - \frac{1}{5}x + \frac{1}{3}\sqrt{25+x^2}$$

critical # at $x = \frac{15}{4}$

y-values:

$t\left(\frac{15}{4}\right) = \ldots = \frac{10}{3}$ ← smallest!

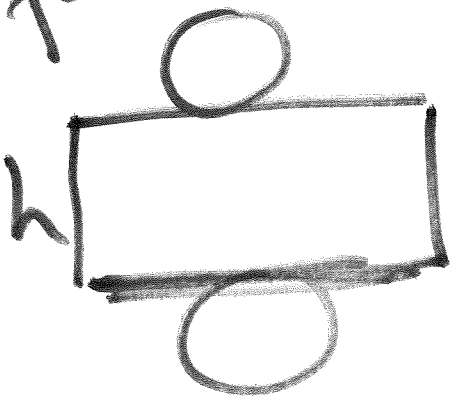
$t(0) = \frac{11}{3}$

$t(10) = \ldots \frac{5\sqrt{5}}{3} > \frac{5 \cdot \sqrt{4}}{2} = \frac{5 \cdot 2}{3}$

Volume of cylinder

$V = \pi r^2 h$

surface area of cylinder



$S = 2\pi r^2 + 2\pi r h$

Worksheet: Optimization Problems A

Problem 1: You have been asked to design a can shaped like a right cylinder using ~~400~~ A cm^2 of material. What dimensions will result in the largest volume?

1. **IDENTIFY WHAT QUANTITY NEEDS TO BE OPTIMIZED** maximize volume
2. **SKETCH** and 3. **LABEL VARIABLES**

r, h

4. **WRITE DOWN A FUNCTION** REPRESENTING THE QUANTITY THAT NEEDS TO BE OPTIMIZED.
   How many variables does your function have? 2
   If more than one variable, then we need to ***reduce the number of variables*** *using constraints.*

$V = \pi r^2 h$

5. **IDENTIFY CONSTRAINTS.** In this problem the constraint is surface area = 400

$A \quad 400 = 2\pi r^2 + 2\pi r h \hookrightarrow h = \left(\frac{400 - 2\pi r^2}{2\pi r}\right)$

6. **REWRITE FUNCTION** IDENTIFIED IN 4.

$V = \pi r^2 \left(\frac{400 - 2\pi r^2}{2\pi r}\right) = r\left(\frac{200}{2} - \pi r^2\right)$

7. **IDENTIFY THE DOMAIN** OF THE FUNCTION

$h \geq 0, \; r \geq 0$ and $\frac{400 - 2\pi r^2}{2\pi r} \geq 0$

8. **USE CALCULUS** to find global extremum.

(homework)

$V(r) = r(200 - \pi r^2)$

$0 \leq r \leq \sqrt{\frac{200}{\pi}}$

$(400 - 2\pi r^2) \geq 0$

$r = \sqrt{\frac{400}{2\pi}}$ 0 $\sqrt{\frac{400}{2\pi}}$ (+) −

$0 < r \leq \sqrt{\frac{400}{2\pi}} = \sqrt{\frac{200}{\pi}}$

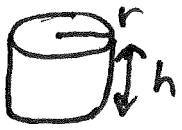
Problem 2

How would your answer change if the can had to have a surface area of a cm^2.

Worksheet: Optimization Problems

Problem 1: You have been asked to design a can shaped like a right cylinder using 400 cm^2 of material. What dimensions will result in the largest volume?

1. **IDENTIFY WHAT QUANTITY NEEDS TO BE OPTIMIZED :** maximize volume of cylinder
2. **SKETCH** and 3. **LABEL VARIABLES**



4. **WRITE DOWN A FUNCTION** REPRESENTING THE QUANTITY THAT NEEDS TO BE OPTIMIZED.

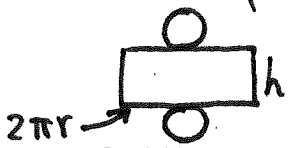
$V = \pi r^2 h$

How many variables does your function have? 2

If more than one variable, then we need to ***reduce the number of variables*** *using constraints.* → need 1 constraint

5. **IDENTIFY CONSTRAINTS.** In this problem the constraint is amount of material (400 m^2)

surface area of cylinder = 400



$400 = 2(\pi r^2) + 2\pi r \cdot h \rightarrow h = \frac{400 - 2\pi r^2}{2\pi r}$

6. **REWRITE FUNCTION** IDENTIFIED IN 4.

$V(r) = \pi r^2 \cdot \left(\frac{400 - 2\pi r^2}{2\pi r}\right) = r \cdot \frac{2(200 - \pi r^2)}{2} = 200r - \pi r^3$

7. **IDENTIFY THE DOMAIN** OF THE FUNCTION

$r \geq 0$

$h \geq 0 \rightarrow \frac{400 - 2\pi r^2}{2\pi r} \geq 0$ → $200 - \pi r^2 \geq 0$ so $0 \leq r \leq \sqrt{\frac{200}{\pi}}$

$r = \sqrt{\frac{200}{\pi}}$ (reject negative sol.)

8. **USE CALCULUS** to find global extremum.

$V'(r) = 200 - 3\pi r^2 = 0$

$r = \sqrt{\frac{200}{3\pi}}$

$V\left(\sqrt{\frac{200}{3\pi}}\right) = 200\sqrt{\frac{200}{3\pi}} - \pi\left(\sqrt{\frac{200}{3\pi}}\right)^3$

$V(0) = 0$

$V\left(\sqrt{\frac{200}{\pi}}\right) = 200\sqrt{\frac{200}{\pi}} - \pi\left(\sqrt{\frac{200}{\pi}}\right)^3$

hard to compare y-values without calculator → use other method

use 2nd derivative test

$V''(r) = -6\pi r$

$V''\left(\sqrt{\frac{200}{3\pi}}\right) = -6\pi\sqrt{\frac{200}{3\pi}} < 0$

so there is a loc. min at $r = \sqrt{\frac{200}{3\pi}}$. Since there is only one critical number, the local minimum must be a global minimum.

Problem 2

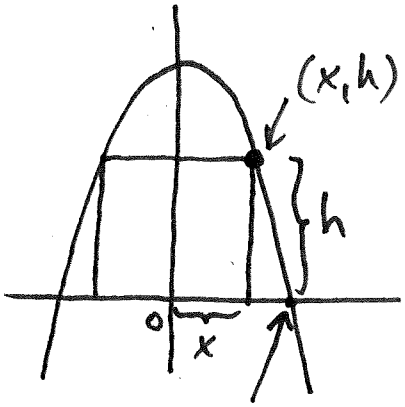
How would your answer change if the can had to have a surface area of a cm^2.

constraint is $a = 2\pi r^2 + 2\pi r h$

so $h = \frac{a - 2\pi r^2}{2\pi r}$

$V(r) = \pi r^2\left(\frac{a - 2\pi r^2}{2\pi r}\right) = r\left(\frac{a}{2} - \pi r^2\right)$ ← maximize this keeping in mind that a is constant

Problem 3: Find the dimensions of the rectangle of maximum area that can be inscribed in the region bounded by the x-axis and the parabola $y = k - x^2$, where k is a constant.



need to maximize area of rectangle

$A = (2x) \cdot h$ ← 2 variables, need 1 constraint

constraint: top vertices of rectangle must lie on parabola so

$h = k - x^2$

Then $A(x) = (2x)(k - x^2)$ for $0 \le x \le \sqrt{k}$

$\sqrt{k}$ because $k - x^2 = 0$, $x = \pm\sqrt{k}$

$\boxed{A(x) = 2kx - 2x^3}$, k is a parameter does not depend on x.

$A'(x) = 2k - 6x^2 = 0$

$6x^2 = 2k$

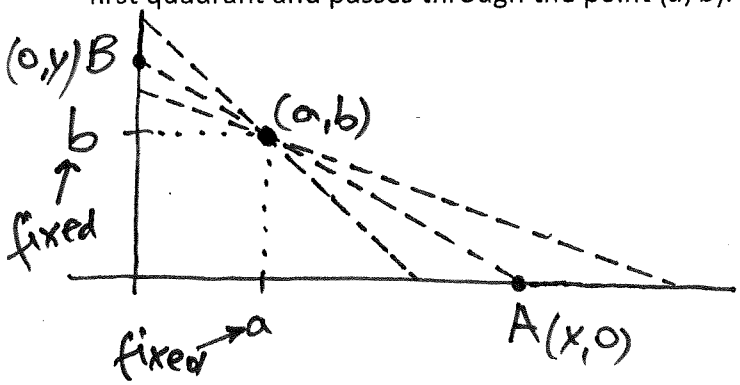
$x = \sqrt{\frac{k}{3}}$, $x = -\sqrt{\frac{k}{3}}$ ↑ not in domain reject.

To prove there is a global max at critical number we use 2nd derivative test (easiest)

$A''(x) = -12x < 0$ for all x

so $A\left(\sqrt{\frac{k}{3}}\right)$ is global maximum

Problem 4: Let a and b be positive numbers. Find the length of the shortest line segment that is cut off by the first quadrant and passes through the point (a, b).



need to minimize length of segment AB.

Let $A(x,0)$ and $B(0,y)$, then

$L = \sqrt{x^2 + y^2}$ ← 2 variables, need 1 constraint

constraint: segment goes through (a,b) then

slope of segment $m = \frac{0-y}{x-0} = \frac{0-b}{x-a} \rightarrow -\frac{y}{x} = \frac{-b}{x-a}$

so $y = \frac{bx}{x-a}$. Note $y > b$ and $x > a$. Endpoints not include.

need to minimize $L(x) = \sqrt{x^2 + \left(\frac{bx}{x-a}\right)^2}$ for $x > a$

could write $L^2 = x^2 + \frac{b^2x^2}{(x-a)^2}$ and differentiate implicitly to find $L'(x)$

$$2LL' = 2x + \frac{b^2 \cdot 2x(x-a)^2 - b^2x^2 \cdot 2(x-a)}{(x-a)^4}$$

$$L' = \frac{1}{2L}\left(2x + \frac{2b^2x(x-a)(x-a-x)}{(x-a)^4}\right)$$

$$L' = \frac{1}{2L}\left(2x + \frac{2b^2x\cancel{(x-a)}(-a)}{(x-a)\cancel{^4}3}\right) = 0$$

$$2x(x-a)^3 - 2ab^2x = 0$$

$$2x\left[(x-a)^3 - ab^2\right] = 0$$

$x=0$
not in domain
reject

$$(x-a)^3 - ab^2 = 0$$
$$(x-a)^2 = ab^2$$
$$x-a = \sqrt[3]{ab^2}$$
$$x = a + \sqrt[3]{ab^2}$$

Compare y-value with asymptotic behaviour

$$\lim_{x\to a^+} L(x) = \lim_{x\to a^+} \sqrt{x^2 + \underbrace{\frac{b^2x^2}{(x-a)^2}}_{\to 0 \to \infty}} = \infty$$

$$\lim_{x\to+\infty} L(x) = \lim_{x\to\infty} \sqrt{\underbrace{x^2}_{\to\infty} + \underbrace{\frac{b^2x^2}{(x-a)^2}}_{\to b^2}} = \infty$$

because $\dfrac{b^2x^2}{x^2-2ax+a^2} = \dfrac{b^2}{1-\frac{2a}{x}+\frac{a^2}{x^2}}$ ($\frac{2a}{x} \to 0$, $\frac{a^2}{x^2} \to 0$)

so $L\left(a+\sqrt[3]{ab^2}\right)$ is global minimum.