Nov 20

## Overview

- Review of Differentiation
- Derivative of $\ln[g(x)]$
- Derivative of $a^x$ for $a>0$
- Growth Models

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### 1. Review of Differentiation.

**Example 1** Find the derivative of

$$f(x) = 3\sqrt{x} + 4\ln x.$$

$$\frac{df(x)}{dx} = 3\cdot\frac{1}{2}x^{-\frac{1}{2}} + 4\cdot\frac{1}{x}$$

$$= \frac{3}{2\sqrt{x}} + \frac{4}{x}$$

**Example 2** Differentiate $f(x) = \underbrace{x^2}_{g(x)}\cdot\underbrace{\ln x}_{h(x)}$.

$$\frac{d}{dx}[g(x)\,h(x)] = g'(x)\,h(x) + g(x)\,h'(x)$$

$$\frac{df}{dx} = 2x\cdot\ln x + x^2\cdot\frac{1}{x}$$

$$= 2x\cdot\ln x + x$$

## 2. Derivative of $\ln[g(x)]$

$$\boxed{\frac{d}{dx}\ln x = \frac{1}{x}}$$

$$\frac{d}{dx}\ln[g(x)]$$

$$\frac{d}{dx}f(g(x)) = f'(g(x)) \cdot g'(x)$$

$$\frac{df}{d(g(x))}$$

$$g(x) = g(x) \qquad f(x) = \ln x$$

$$\frac{d}{dx}\ln[g(x)] = \frac{d}{dx}f(g(x))$$

$$= f'(g(x)) \cdot g'(x)$$

$$= \frac{1}{g(x)} \cdot g'(x)$$

$$= \frac{g'(x)}{g(x)}$$

$$\boxed{\frac{d}{dx}\ln[g(x)] = \frac{g'(x)}{g(x)}}$$

**Example 3** $\frac{d}{dx}\left[\ln(\underbrace{x^2+1}_{g(x)})\right]$

$$= \frac{2x}{x^2+1}$$ ✓

**Example 4** Differentiate $f(x) = \ln(\underbrace{\sqrt{x}+1}_{g(x)})$

$$\frac{df}{dx} = \frac{\frac{1}{2}\cdot x^{-\frac{1}{2}} + 0}{\sqrt{x}+1}$$

$$= \frac{1}{2(\sqrt{x}+1)\sqrt{x}}$$

## 3. Derivative of $a^x$ for $a>0$.

$$\frac{d}{dx}e^x = e^x$$

$$\frac{d}{dx}e^{3x} = e^{3x}\cdot 3$$

$$\frac{d}{dx}2^x = ?$$

$$\frac{d}{dx} a^x$$

$$a = e^{\ln a}$$

$$a^x = (e^{\ln a})^x$$

↓ property (C. of Exponent Function

$$= e^{(\ln a \cdot x)}$$

↑ constant/number

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{\ln a \cdot x}$$

↓ chain Rule

$$= \ln a \cdot e^{\ln a \cdot x}$$

$$= \ln a \cdot a^x$$

$$\boxed{\frac{d}{dx} a^x = \ln a \cdot a^x}$$

$a = e$ $\quad \frac{d}{dx} e^x = \ln e \cdot e^x$

$$= 1 \cdot e^x = e^x$$

**Example 5** $\frac{d}{dx} 2^x = \ln 2 \cdot 2^x$

**Example 6** Differentiate

$$y = \ln(\underbrace{x^2 - 3x \cdot 2^x}_{g(x)}) \quad *$$

$$\frac{dy}{dx} = \frac{g'(x)}{g(x)}$$

$$= \frac{(x^2 - 3x \cdot 2^x)'}{x^2 - 3x \cdot 2^x}$$

$$= \frac{2x - (3x \cdot 2^x)'}{x^2 - 3x \cdot 2^x}$$

$$= \frac{2x - [3 \cdot 2^x + 3x \cdot \ln 2 \cdot 2^x]}{x^2 - 3x \cdot 2^x}$$

# 4. Growth Models

## • population growth.

**Example 1** A certain bacteria doubles in size every 10 hrs. Given that there was only 1 bacterium to start with, how many bacteria will be there in 1 day?

| Time / hrs | # of Bacteria |
|---|---|
| T = 0 | $1 = 2^{0}$ |
| T = 10 | $2 = 2^{1}$ |
| T = 20 | $4 = 2^{2}$ |
| T = 30 | $8 = 2^{3}$ |
| T = 40 | $16 = 2^{4}$ |
| ⋮ | ⋮ |
| $T = t$ | $2^{t/10}$ |

If we start with 1 bacterium, at time $t$, the # of bacteria is

$$2^{t/10} = (e^{\ln 2})^{t/10}$$

$2 = e^{\ln 2}$

$$= e^{\frac{\ln 2 \cdot t}{10}}$$

$$= e^{\frac{\ln 2}{10} \cdot t}$$

1 day = 24 hrs

$t = 24$, # of bacteria is $2^{24/10} = e^{\frac{\ln 2}{10} \cdot 24}$

$$\approx 5.28.$$

Wed Nov 22

Today: exponential models.

Growth models.

$f(t)$ = quantity
↑ time

i.e. size of population of fish in a lake.

$f'(t)$ = rate of change of your quantity at time $t$

$\frac{df}{dt}$

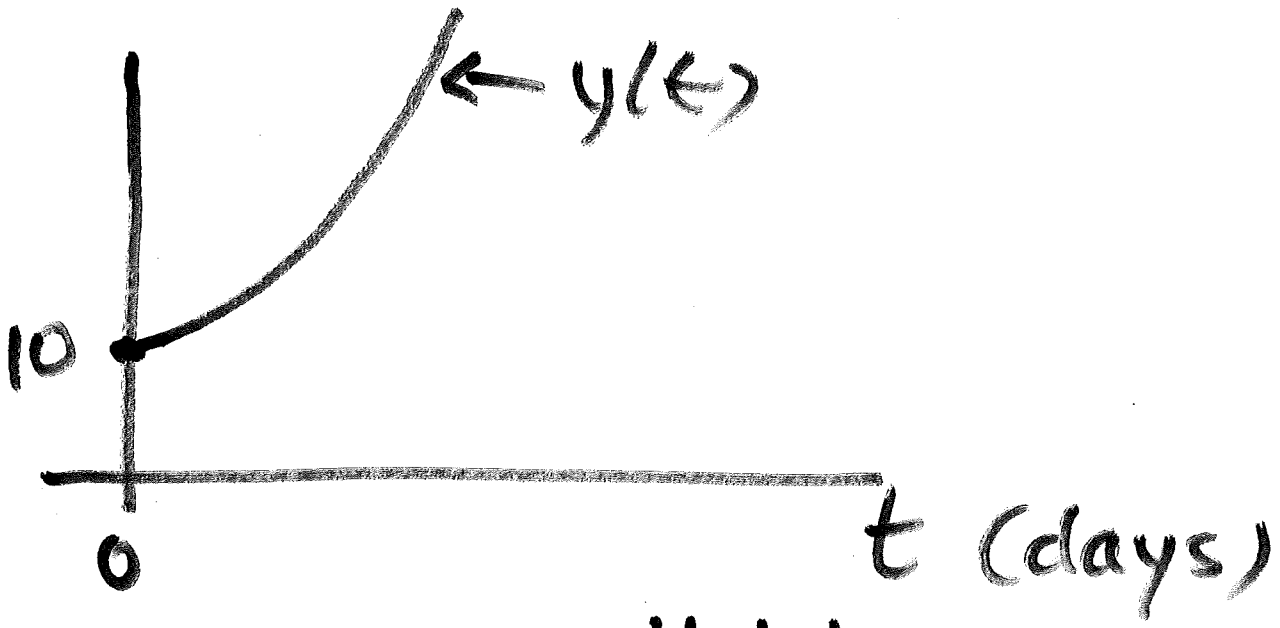
ex: Suppose a bacteria culture contains $y$ bacteria. and $\boxed{y(t) = 10e^{3t}}$ is the # of bacteria $t$ days after you started the culture.

How many bacteria did you have at the start?

$y(0) = 10 \cdot e^{3.0} = 10 \cdot e^{0} = 10$ bact.

after t=10 days

$y(10) = 10 \cdot e^{3 \cdot 10} = 10e^{30}$ a lot!




How long will it take the population to double in size?

$y(0) = 10$, find t when $y = 20$

Solve for $t$

$$10e^{3t} = \frac{20}{10}$$

$$e^{3t} = 2$$

take logs.

$$\log(e^{3t}) = \log(2)$$

$$3t = \log(2)$$

$$t = \frac{\log(2)}{3} = \frac{\ln(2)}{3}$$

doubling time

$\approx 0.23$ days.

Does the doubling time change over time? ← NO!

suppose $y(0) = y_0$ ← fixed

let $t_d$ be the time it takes to double in size

Suppose at time $t$, $y(t) = Ce^{kt}$ where $C$ and $k$ are constant (fixed numbers)

Observation 1:

$$y(0) = y_0 = C \cdot e^{k \cdot 0} = C \cdot 1$$

$$\Rightarrow C = y_0 \leftarrow \text{initial value}$$

so $\boxed{y(t) = y_0 e^{kt}}$

Observation 2: $y(t_d) = 2y_0$

$$\Rightarrow y(t_d) = y_0 e^{k \cdot t_d} = 2y_0$$

$$\frac{y_0}{y_0} e^{kt_d} = \frac{2y_0}{y_0}$$

$$e^{kt_d} = 2$$

$$kt_d = \ln(2)$$

doubling time $t_d = \dfrac{\ln(2)}{k}$ ← does not depend on $t$

**Remark 1** if a quantity grows exponentially over time, its doubling time is a constant (fixed over time) it depends only on constant $k$

What does $k$ represent?

$$y(t) = y_0 e^{kt}$$

Observation 3: $y'(t) = y_0 (e^{kt})' =$

$$= (y_0 e^{kt}) k$$

$\uparrow y(t)$

$$\Rightarrow y' = k y$$

$y'$: rate of change of quantity of interest

$y$: size of quantity of interest

**Remark 2** if a quantity grows exponentially over time, its instantaneous rate of change (growth rate) is proportional to the size of the quantity. $y' = ky$

the constant of proportionality is $k$.

$$k = \frac{y'}{y} = \text{relative growth rate.}$$

$k$ fixed over time

Friday Nov 24

Today: examples of exponential models (growth and decay)

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Last time:

If a quantity $y(t)$ grows

- with a constant (fixed) doubling time

or

- at a constant (fixed) relative growth rate $k$

then we can model the growth of the quantity $y(t)$ as

$y' = ky$ and $\boxed{y(t) = y(0)e^{kt}}$

Examples:

**Problem 1**: The number of cells in a tumor doubles every 6 weeks, starting with 8 cells. After how many weeks does the tumor have 1500 cells?

constant doubling time
$\Rightarrow p(t) = p(0)e^{kt}$
where $p(t)$ = # of cells at $t$ weeks

we know $p(0) = 8$
$\Rightarrow p(t) = 8e^{kt}$ need to find $k$

we know tumor doubles in size in 6 weeks
so if $p(0) = 8 \Rightarrow p(6) = 2 \cdot 8 = 16$
then $p(6) = 8 \cdot e^{k \cdot 6} = 16$ solve for $k$

$$e^{6k} = 2$$
$$6k = \ln(2)$$
$$k = \frac{\ln(2)}{6}$$

Warning: $\frac{\ln(2)}{6} \neq \ln(\frac{2}{6})$

need to find at what $t$ is $p = 1500$

given $p(t) = 8e^{\left(\frac{\ln 2}{6}\right)\cdot t}$

$$1500 = 8e^{\left(\frac{\ln 2}{6}\right)t}$$

$$\frac{1500}{8} = \frac{375}{2} = e^{\left(\frac{\ln 2}{6}\right)t}$$

$$\ln\left(\frac{375}{2}\right) = \left(\frac{\ln 2}{6}\right)\cdot t$$

$$t = \frac{\ln\left(\frac{375}{2}\right)}{\frac{\ln(2)}{6}} = \frac{6\ln\left(\frac{375}{2}\right)}{\ln(2)} \sim 45.3 \text{ weeks}$$

Warning: do <u>not</u> cancel out the logs here!

**Problem 2:** In 1927 the population of the world was about 2 billion. In 1974 it was about 4 billion.

① Estimate when it reached 6 billion.

② What will the population of the world be in 2100, assuming it grows exponentially?

means $y(t) = y(0)\,e^{kt}$

where $y(t)$ = population at $t$ yr.

set $t=0$ in 1927 $\Rightarrow y(0) = 2$

$t=47$ in 1974 $\Rightarrow y(47) = 4$

we find $k$ from

$$y(47) = y(0)\,e^{k\cdot 47} = 2e^{k\cdot 47} = 4$$

$$e^{47\cdot k} = 2$$

$$47k = \ln(2)$$

$$k = \frac{\ln(2)}{47} \approx 0.014 = 1.4\%$$
(relative growth rate)

① we find $t$ when $y(t) = 6$ given

$$y(t) = 2e^{\left(\frac{\ln(2)}{47}\right)t}$$

$$6 = 2e^{\frac{\ln 2}{47} \cdot t}$$

$$3 = e^{\frac{\ln 2}{47} \cdot t}$$

$$\ln 3 = \frac{\ln 2}{47} \cdot t$$

$$t = \frac{\ln 3}{\frac{\ln 2}{47}} = \frac{47 \ln 3}{\ln 2} \approx 74.5$$

calendar year 2001 (1927 + 74)

② we find $p(173)$ because $2100 - 1927 = 173$

$$p(173) = 2e^{\frac{\ln 2}{47} \cdot 173}$$

$\approx 26$ billion!

↑ unsustainable!

# Exponential Decay

If a quantity $y(t)$ decays at a rate proportional to the size of the quantity, we have

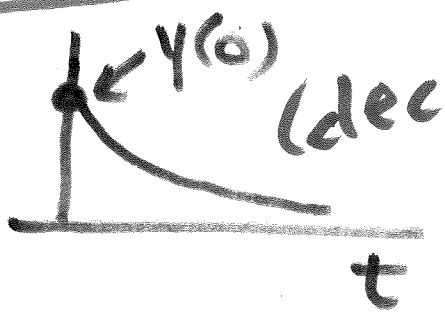
$$y' = -ky \quad \text{where } k > 0$$

$k$ is constant over time (fixed)

and

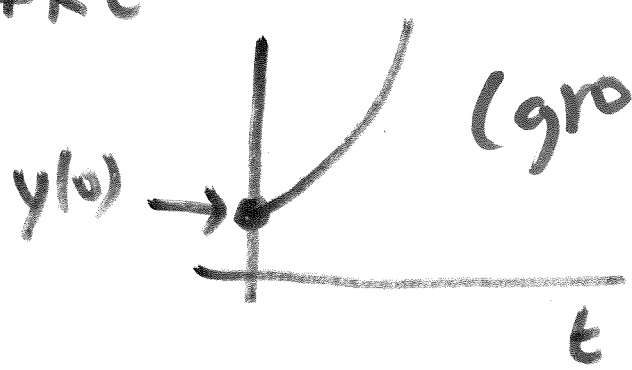
$$\boxed{y(t) = y(0)e^{-kt}}$$

graph of $y(t)$




(compare to exponential growth

$$y(t) = y(0)e^{+kt}$$



**Problem 3** A population of rabbits has been infected with a virus. The animals are dying at a rate of 2% per month.

If there were 2000 rabbits initially how long does it take for the population to reduce its size by half?

constant decay rate

$\Rightarrow p(t) = p(0)e^{-kt}$

where $p(t)$ = # of rabbits at $t$ weeks.

we know $p(0) = 2000$

need to find $t$ when $p(t) = \frac{1}{2} \cdot 2000 = 1000$

given that $p(t) = 2000e^{-kt}$

need to find $k$.

we know

constant decay % rate $k = \frac{y'}{y} = 2\% = 0.02$

So $p(t) = 2000\, e^{-0.02 \cdot t}$

So $1000 = 2000\, e^{-0.02 \cdot t}$

$\frac{1}{2} = e^{-0.02 \cdot t}$

$\ln\left(\frac{1}{2}\right) = -0.02\, t$

$t = \frac{\ln(1/2)}{-0.02} \approx 35$ weeks

Note : $\ln(1/2) < 0$

$\ln(1/2) = \underbrace{\ln(1)}_{0} - \ln(2) = -\ln(2)$

So $t = \frac{\ln(1/2)}{-0.02} > 0$

(neg. time wouldn't make sense!)