In the following questions, justify your answers and show all your work. Unless otherwise indicated, simplification of answers are required.

1. This question has two independent problems with multiple parts.

4 marks
(a) Consider the function $f(x)$ whose graph is shown below.

Evaluate the following limits or determine they either do not exist or approach (positive or negative) infinity.

(i) $\lim_{x \to 0^+} f(x) = 4 \quad \text{WHY?}$

(ii) $\lim_{x \to 6} f(x) \quad \text{DNE} \quad \text{WHY?}$

(iii) $\lim_{x \to 3} \frac{1}{f(x) - 2} = +\infty \quad \text{WHY?}$

(iv) $\lim_{h \to 0} \frac{f(7 + h) - f(7)}{h} = 1 \quad \text{WHY?}$

4 marks
(b) Evaluate the following limits or determine they either do not exist or approach (positive or negative) infinity.

(i) $\lim_{x \to -2} (x^2 + 5x + 6) = (-2)^2 + 5(-2) + 6 = 4 - 10 + 6 = 0$

(ii) $\lim_{x \to -2} \frac{x^2 + 5x + 6}{x^2 + 3x + 2} = \lim_{x \to -2} \frac{(x+2)(x+3)}{(x+2)(x+1)} = \frac{-2+3}{-2+1} = -1$

WHY DO WE FACTOR AND CANCEL $(x+2)$?

WHY IS $0/0$ NOT AN ACCEPTABLE ANSWER?

(iii) $\lim_{t \to 2} \frac{t^3}{8 - t^3} = +\infty \quad \text{WHY?}$

WHY IS IT NOT SUFFICIENT TO SAY DNE?
2. This question has two independent problems.

(a) (i) Let \( f(x) = \frac{x}{x+3} \) and \( g(x) = \sqrt{1-x} \). Find the domain of \( g(f(x)) \).

\[
g\left( f(x) \right) = \sqrt{1 - \frac{x}{x+3}} = \sqrt{\frac{3}{x+3}}
\]

\text{Domain:} \quad \frac{3}{x+3} > 0

\quad \implies x + 3 > 0 \quad \text{\textit{why do we set}} \quad x + 3 > 0? \quad \text{\textit{why not}} \quad x + 3 \geq 0?

\quad \implies x > -3

Let (ii) \( u(x) = 3x + 2 \) and \( v(x) = 2x + A \). Find \( A \) such that \( u(v(x)) = v(u(x)) \).

\[
u(v(x)) = 3(2x + A) + 2 \quad \text{\textit{why?}}
\]

\[
v(u(x)) = 2(3x + 2) + A \quad \text{\textit{why?}}
\]

\[
\text{So} \quad 3(2x + A) + 2 = 2(3x + 2) + A
\]

\[
6x + 3A + 2 = 6x + 4 + A
\]

\[
2A = 2
\]

\[
A = 1
\]
3. Suppose the temperature $T(t)$ (in degrees °C) in this room at $t$ minutes after the start of the exam has been recorded and reported in the table below.

<table>
<thead>
<tr>
<th>$t$</th>
<th>0</th>
<th>5</th>
<th>7</th>
<th>11</th>
<th>15</th>
<th>19</th>
<th>21</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T(t)$</td>
<td>18</td>
<td>18.2</td>
<td>18.5</td>
<td>18.9</td>
<td>19.2</td>
<td>19.2</td>
<td>19.2</td>
</tr>
</tbody>
</table>

(a) Suppose we invert the function $T$ on the interval [0, 11] and construct the inverse function $T^{-1}$. Explain the meaning of the output generated by $T^{-1}$.

The output of $T^{-1}$ is time for a specific temperature.

E.g. If $T(5) = 18.2$ then $T^{-1}(18.2) = 5 \Rightarrow$ output is time

Why is the output "time"?

(b) Explain why based on the data available we can claim that the temperature in the room was exactly 19 °C at some point in time.

Why do we need this observation?

We observe $T(11) = 18.9 < 19 < 19.2 = T(15)$

$T(t)$ must be a [continuous] function on [11, 15].

Therefore by IVT there must be a time $c$ in [11, 15] when $T(c) = 19$.

Why do we need to specify that $T$ is a continuous function?

(c) Estimate $T'(19)$. Provide a rationale for your estimate.

Since $T(t)$ does not change over the interval [15, 21], we estimate $T'(19) = 0$

Why zero?

(d) Estimate at what rate is the temperature in the room changing at $t = 7$.

Rate of change of $T$ at $t = 7$ is approx. $\frac{T(11) - T(7)}{11 - 7}$

Why do we estimate the rate of change like this?

Is this the only way to estimate the rate of change of $T$ at $t = 7$?
4. (a) Find all values of \( x \) where \( H(x) \) is continuous, where

\[
H(x) = \begin{cases} 
\frac{x^2 + 5x + 6}{x + 3}, & x < 0 \\
\frac{x^2}{x^3 + x^2}, & x > 0
\end{cases}
\]

Note \( H(x) \) is not defined at \( x = 0 \), so \( H(x) \) is not continuous at \( x = 0 \).

Also \( H(x) \) is defined piecewise. Each "piece" is a rational function. A rational function is continuous at any point where it is defined.

For \( x < 0 \), \( \frac{x^2 + 5x + 6}{x + 3} \) is defined for all \( x \) such that
\[
(x + 3 \neq 0, x \neq -3)
\]

For \( x > 0 \), \( \frac{x^2}{x^3 + x^2} \) is defined for all \( x \) such that
\[
(h(x) = 0, x^3 + x^2 \neq 0)
\]

\( H(x) \) is continuous for all \( x \neq 0 \) and \( x \neq -3 \).

Why is it not necessary to evaluate \( \lim_{x \to 0} H(x) \) and \( \lim_{x \to 0^+} H(x) \)?

(b) Find \( a \) and \( b \) such that the following function is continuous everywhere

\[
g(y) = \begin{cases} 
\sqrt{3} - y, & y < -1 \\
ay + b, & -1 \leq y \leq 1 \\
10y - 8, & y > 1
\end{cases}
\]

Each "piece" is continuous on its interval of definition.

Check boundary points.

At \( y = -1 \), \( g(y) \) is continuous if
\[
\lim_{y \to -1^-} g(y) = \lim_{y \to -1^+} g(y) = g(-1)
\]

\[
\lim_{y \to -1^-} (\sqrt{3} - y) = \sqrt{4} = 2,
\lim_{y \to -1^+} (ay + b) = a(-1) + b = -a + b = g(-1)
\]

so \( 2 = -a + b \).

At \( y = 1 \), \( g(y) \) is continuous if
\[
\lim_{y \to 1^-} g(y) = \lim_{y \to 1^+} g(y) = g(1)
\]

\[
\lim_{y \to 1^-} (ay + b) = a + b = g(1),
\lim_{y \to 1^+} (10y - 8) = 2
\]

so \( a + b = 2 \).

\[-a + b = 2 \Rightarrow a = 0, b = 2\]
(c) Find a valid function \( L(t) \) such that the following function \( F(t) \) is continuous everywhere:

\[
F(t) = \begin{cases} 
3^t, & t < 0 \\
L(t), & 0 \leq t \leq 5 \\
\frac{1}{t+3}, & t > 5
\end{cases}
\]

Try. \( L(t) = at + b \) \( \text{ IS THIS THE ONLY OPTION?} \)

At \( t = 0 \), to make \( F(t) \) continuous we need \( \lim_{t \to 0} F(t) = F(0) \)

\[
\lim_{t \to 0^-} (3^t) = \lim_{t \to 0^+} (at + b) = b
\]

\( \text{WHY } 3^t? \)

\( i = b \)

\( \text{WHY } i? \)

At \( t = 5 \), we need \( \lim_{t \to 5} F(t) = F(5) \)

\[
\lim_{t \to 5^-} (at + b) = \lim_{t \to 5^+} \left( \frac{1}{t+3} \right) = \frac{1}{5+3} (a \cdot 5 + b)
\]

\( 5a + b = \frac{1}{8} \)

\( \text{WHY } \frac{1}{t+3}? \)

\( 5a + 1 = \frac{1}{8} \)

\( a = -\frac{3}{40} \)
5. Consider the function \( f(t) = 3t^2 + t \) for \( t \geq 0 \).

Let \( P(t_P, y_P) \) and \( Q(t_Q, y_Q) \) be two points on the graph of \( f \) with \( t_P = 2 \) and \( t_Q = 2 + h \), for \( h \neq 0 \).

(a) Find the slope \( m_{PQ} \) of the line that goes through \( P \) and \( Q \). Simplify your answer.

\[
\begin{align*}
\text{Why do we compute this?} \\
m_{PQ} &= \frac{f(2+h) - f(2)}{2+h - 2} \\
&= \frac{3(2+h)^2 + (2+h) - (3 \cdot 2^2 + 2)}{h} \\
&= \frac{3(4 + 4h + h^2) + 2 + h - 14}{h} \\
&= \frac{14 + 13h + 3h^2 - 14}{h} \\
&= \frac{13 + 3h}{h} \quad \text{Why are we allowed to cancel } h \text{ here?} \\
&= 13 + 3 \\
\end{align*}
\]

(b) Compute \( \lim_{h \to 0} m_{PQ} \) and explain what this limit represents graphically.

\[
\lim_{h \to 0} m_{PQ} = \lim_{h \to 0} (13 + 3h) = 13
\]

This is the slope of the tangent line at \( (2, 14) \).

(c) Suppose \( f(t) \) represents the position (in m) of a car at time \( t \) (in s).

(i) If \( h = 1 \), what does \( m_{PQ} \) represent in this case?

\[
m_{PQ} = \frac{f(3) - f(2)}{3 - 2} = \frac{\text{change in position}}{\text{change in time}}
\]

This is the average velocity of the car over \( (2, 3) \).

Why is it not sufficient to say just "velocity" here?

(ii) What is the instantaneous velocity of the car at \( t = 2 \)?

\[
u(2) = \lim_{h \to 0} \frac{f(2+h) - f(2)}{h} = 13
\]

Why 13?

(iii) At \( t = 2 \), is the car moving away from or towards its initial position? Explain.

At \( t = 2 \), \( v(2) = 13 > 0 \) so position is increasing, therefore the car is moving away from its initial position.
6. This question has three independent problems. You may use differentiation rules to compute derivatives as appropriate.

(a) Find \( y' \) if \( y = 2x^5 + \sqrt[3]{x^2} - \frac{3}{\sqrt[5]{x^3}} + 120^{-1/5} \).

\[
y' = 2 \cdot 5 \cdot x^4 + \frac{2}{3} \cdot x^{-1/3} - 3 \cdot \left(-\frac{1}{5}\right) \cdot x^{-4/5}
\]

(b) For what value(s) of \( x \) does the graph of \( f(x) = x^3 + 3x^2 + x + 3 \) have a horizontal tangent line?

\[
f'(x) = 3x^2 + 6x + 1 = 0
\]

\[
x = \frac{-6 \pm \sqrt{36 - 4 \cdot 3 \cdot 1}}{6} = \frac{-6 \pm \sqrt{24}}{6} = \frac{-6 \pm 2\sqrt{6}}{6} = \frac{-3 \pm \sqrt{6}}{3}
\]

(c) Find an equation of the line tangent to the graph of \( y = 1 + 4x + x^2 \) at the point of coordinate \( x = 1 \).

\[
y' = 4 + 2x
\]

\[
y'(1) = 4 + 2 \cdot 1 = 6 \leq 6
\]

\[
y(1) = 1 + 4 \cdot 1 + 1^2 = 6 \leq 6 (x - 1)
\]

\[
tangent\ line\ at\ (1,6)\ is\ y - 6 = 6(x - 1) \leq y = 6x
\]
7 marks 7. (a) Sketch the graph of a continuous function \( f \) defined on \([0, 1]\) and with range \([0, 1]\). Make sure you label the coordinate axes. Your sketch must be to scale.

(b) Draw the line \( y = x \) on the same diagram you drew in part (a). Based on your sketch, how many intersections does your function \( f \) have with the line \( y = x \), that is, how many solutions does \( f(x) = x \) have?

My function intersects the line \( y = x \) once, so \( f(x) = x \) has one solution.

(c) Using appropriate theorem(s) discussed in class, prove that any function \( f \) defined on \([0, 1]\) and with range \([0, 1]\) must have at least one point \( x \) in its domain such that \( f(x) = x \). (Hint: Consider the function \( F(x) = f(x) - x \).)

Consider \( F(x) = f(x) - x \). This is a continuous function on \([0, 1]\) because both \( f(x) \) and \( x \) are continuous on \([0, 1]\).
Observe \( F(0) = f(0) - 0 = f(0) \geq 0 \) by definition of \( f \).
If \( f(0) = 0 \), then \( x = 0 \) is a solution of \( f(x) = x \).
If \( f(0) \neq 0 \), then \( F(0) > 0 \).
Observe \( F(1) = f(1) - 1 \leq 0 \) by definition of \( f \).
If \( f(1) = 1 \), then \( x = 1 \) is a solution of \( f(x) = x \).
If \( f(1) \neq 1 \), then \( F(1) < 0 \).
Thus, if \( f(0) \neq 0 \) and \( f(1) \neq 1 \), by IVT we have
\[ F(1) < 0 < F(0) \]
By IVT there must be a number \( c \) in \((0, 1)\) such that \( F(c) = 0 \), so \( x = c \) is a solution to \( f(x) = x \).