MATH 110 December Exam, December 5th, 2014
Duration: 150 minutes

*This test has **11 questions** on **13 pages**, for a total of 70 points.*

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ______________________ Last Name: ______________________

Student-No: ______________________ Section: ______________________

Signature: ____

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	5	6	8	6	6	8	7	8	6	6	4	70
Score:												

Full-Solution Problems. In the following questions, justify your answers and **show all your work**. Unless otherwise indicated, **simplification of answers are not required.**

5 marks 1. (a) Give an example of a function whose domain is all real numbers except $x = 1$ and $x = -1$.

[many possible ans. e.g. $y = \frac{1}{(x-1)(x+1)}$]

(b) Give an example of a function whose range is a single number.

[many possible ans. e.g. $y = 3$]

(c) Find the domain of the function $f(x) = \frac{-\sqrt{3x+5}}{|4x^2-1|}$.

[$x \geq -\frac{5}{3}$, $x \neq \pm\frac{1}{2}$]

(d) Consider the function $f(x) = \sqrt{x}$. One of your friends claims that $f(9) = \pm 3$. Give a one sentence argument why your friend's claim is incorrect.

[$f(x) = \sqrt{x}$ is a function so it cannot return two y-values for one x-value.]

$\boxed{\text{6 marks}}$ 2. Determine whether each of the following statements is true or false. If it is true, **provide justification.** If it is false, provide a counterexample. Answers without justification will receive no marks.

(a) $\lim_{x \to 2} \frac{x-2}{x^2-2x} = \frac{1}{2}$. True / False

[True]

(b) The limit $\lim_{x \to 6} \frac{x^2-5x-6}{|x-6|}$ does not exist. True / False

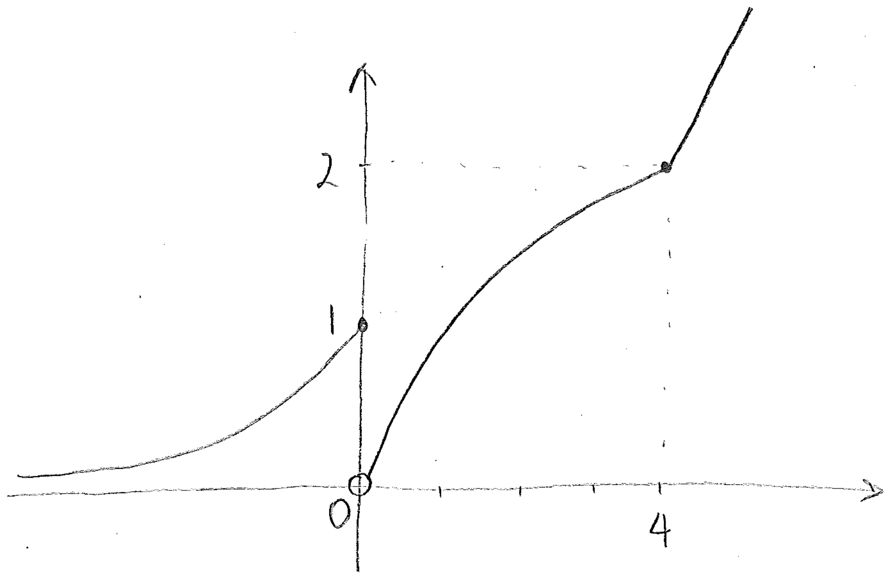
[True]

(c) The limit $\lim_{x \to 2} \frac{f(x)}{g(x)}$ does not exist if $g(2) = 0$. True / False

[False]

8 marks 3. (a) Sketch the graph of the following function,

$$f(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } 0 < x < 4 \\ 2x - 6 & \text{if } x \geq 4. \end{cases}$$



(b) Which one of the following statements is the best reason why $f(x)$ is **not continuous** at $x = 0$?

(i) $f(0)$ does not exist.

(ii) $\lim_{x \to 0} f(x)$ does not exist.

(iii) Both (i) and (ii).

(iv) $f(0)$ and $\lim_{x \to 0} f(x)$ exist, but they are not equal.

[ii]

(c) Which one of the following statements is the best reason why $f(x)$ is **continuous** at $x = 4$?

(i) $f(4)$ exists.

(ii) $\lim_{x \to 4} f(x)$ exists.

(iii) Both (i) and (ii).

(iv) $f(4)$ and $\lim_{x \to 4} f(x)$ exist, and they are equal.

[iv]

(d) Is $f(x)$ differentiable at $x = 4$? Explain why or why not.

No, f(x) is not different, at x=4 because the curve has a corner at x=4

(e) Now consider the function

$$g(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } 0 < x < 4 \\ mx + b & \text{if } x \geq 4. \end{cases}$$

where m and b are constants. Find values for m and b such that $g(x)$ is differentiable at $x = 4$.

$m = \frac{1}{4}$

$b = 1$

6 marks 4. The following limit

$$\lim_{h \to 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{4}}{h}$$

equals the derivative $f'(a)$ for some function f and some point a.

(a) Give an example of one such function f and point a.

$$\left[\begin{array}{l} f(x) = \frac{1}{x^2} \\ a = 2 \end{array}\right]$$

(b) Evaluate the limit above (without using any differentiation rules).

$$\left[-\frac{1}{4}\right]$$

(c) Confirm your answer to part (b) by computing the derivative $f'(a)$ for the function f and point a you identified in part (a) using appropriate differentiation rules.

$$\left[f'(x) = -2x^{-3}\right]$$

6 marks

5. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in metres) above the ground level after t seconds is given by

$$h(t) = 10t - 2t^2.$$

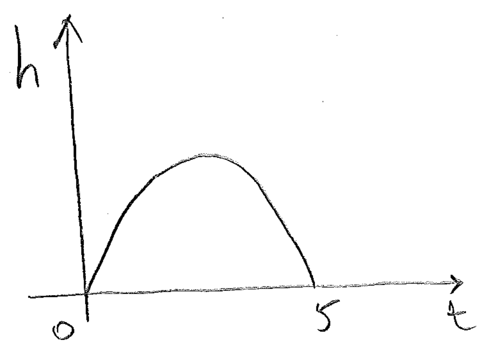
(a) At what time t_h will the rock hit the surface?

$[t_h = 5\,s]$

(b) Find the velocity of the rock when it hits the surface.

$[v(5) = -10 \text{ m/s}]$

(c) Sketch a diagram of the rock's height as a function of time for $0 \le t \le t_h$.



8 marks

6. This question has three independent parts.

(a) Evaluate the following expressions:

$\sin(\frac{2}{3}\pi) = \frac{\sqrt{3}}{2}$ $\cos(\frac{2}{3}\pi) = -\frac{1}{2}$

(b) Find all points where the function $f(x) = x\cos x$ crosses the x-axis.

$x = \frac{\pi}{2} + 2k\pi$

and

$x = \frac{3}{2}\pi + 2k\pi$

(c) Find the equation of the tangent line to the curve $f(x) = x\cos x$ at the point $(\pi, f(\pi))$.

$y = -x$

7 marks

7. Differentiate the following functions. You may leave your answers unsimplified.

(a) $y = (1 + x^2)^{50}$.

$\left[y' = 100x(1+x^2)^{49} \right]$

(b) $f(x) = e^{\sqrt{3x}}$.

$\left[f'(x) = \frac{3}{2\sqrt{3x}} e^{\sqrt{3x}} \right]$

(c) $g(t) = \dfrac{\ln(3t-4)}{t}$.

$\left[g'(t) = \frac{3t - (3t-4)\ln(3t-4)}{t^2(3t-4)} \right]$

8 marks

8. Assume that the number of bacteria at time t (in minutes) follows an exponential growth model $P(t) = Ce^{kt}$, where C and k are constant. Suppose the count in the bacteria culture was 300 after 15 minutes and 1800 after 40 minutes. You may leave your answers unsimplified, i.e. in "calculator-ready" form.

(a) What was the initial size of the culture?

$$\boxed{C = \frac{300}{e^{\frac{3}{5}\ln 6}}}$$

(b) At what rate (in bacteria per minute) was the culture growing initially, i.e. at $t = 0$?

$$\boxed{P'(0) = \frac{300}{e^{\frac{3}{5}\ln 6}} \cdot \frac{\ln 6}{25}}$$

(c) When will the culture contain 3000 bacteria?

$$t = \frac{25}{\ln 6} \ln\left(10e^{\frac{3}{5}\ln 6}\right) \text{ min}$$

$$= \frac{25 \ln 10}{\ln 6} + 15 \text{ min}$$

6 marks

9. When a pebble is thrown into a pond, circular ripples form on the water surface. Suppose $r(t)$ is the radius, in centimetres, of one such ripple at t minutes after the pebble falls into the water, and $A(r)$ is the surface area of a ripple of radius r.

   (a) Which of the following statements best explains the meaning of the composite function $A(r(t))$?

   (i) The radius of the ripple, in centimetres, at time t minutes.

   (ii) The area of the ripple, in square centimetres, of radius r centimetres.

   (iii) The area of the ripple, in square centimetres, at time t minutes.

   (iv) The function f of the minutes and the area.

   (v) The product of the area of the ripple, in square centimetres, by the ripple's radius at time t minutes.

   [iii]

   (b) Suppose the radius of the ripple increases at a rate of 20 centimetres per minute. How fast is the area of the ripple increasing when the radius is 10 cm? Include units. (Recall the area of a circle of radius r is $A = \pi r^2$.)

   $\left[\frac{dA}{dt} = 400\pi \ \frac{cm^2}{min}\right]$

6 marks 10. Find equations of all lines having slope -1 that are tangent to the curve $y = \dfrac{1}{x-1}$.

$y = -x+3$
and
$y = -x-1$

4 marks 11. Show that there is at least one point on the curve $y = x^4 + 2x^2 - 4x - 1$ where the tangent line to the curve is horizontal.

$$y' = 4x^3 + 4x - 4$$

$$4x^3 + 4x - 4 = 0$$

$$x^3 + x - 1 = 0$$

apply IVT