

MATH 110 December Exam, December 5th, 2014

Duration: 150 minutes

This test has 11 questions on 13 pages, for a total of 70 points.

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	9	10	11	Total
Points:	5	6	8	6	6	8	7	8	6	6	4	70
Score:												

Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.
2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.
3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.
4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.
5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
 - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
 - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
 - (iii) purposely viewing the written papers of other examination candidates;
 - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
 - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s)(electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).
6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.
7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.
8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).

Full-Solution Problems. In the following questions, justify your answers and **show all your work**. Unless otherwise indicated, **simplification of answers are not required**.

- 5 marks 1. (a) Give an example of a function whose domain is all real numbers except $x = 1$ and $x = -1$.

$$\left[\begin{array}{l} \text{many possible ans.} \\ \text{e.g. } y = \frac{1}{(x-1)(x+1)} \end{array} \right]$$

- (b) Give an example of a function whose range is a single number.

$$\left[\begin{array}{l} \text{many possible ans.} \\ \text{e.g. } y = 3 \end{array} \right]$$

- (c) Find the domain of the function $f(x) = \frac{-\sqrt{3x+5}}{|4x^2-1|}$.

$$\left[x \geq -\frac{5}{3}, x \neq \pm \frac{1}{2} \right]$$

- (d) Consider the function $f(x) = \sqrt{x}$. One of your friends claims that $f(9) = \pm 3$. Give a one sentence argument why your friend's claim is incorrect.

$$\left[\begin{array}{l} f(x) = \sqrt{x} \text{ is a function} \\ \text{so it cannot return} \\ \text{two } y\text{-values for one} \\ \text{x-value.} \end{array} \right]$$

- 6 marks** 2. Determine whether each of the following statements is true or false. If it is true, provide **justification**. If it is false, provide a counterexample. Answers without justification will receive no marks.

(a) $\lim_{x \rightarrow 2} \frac{x-2}{x^2-2x} = \frac{1}{2}$. True / False

[True]

(b) The limit $\lim_{x \rightarrow 6} \frac{x^2 - 5x - 6}{|x - 6|}$ does not exist. True / False

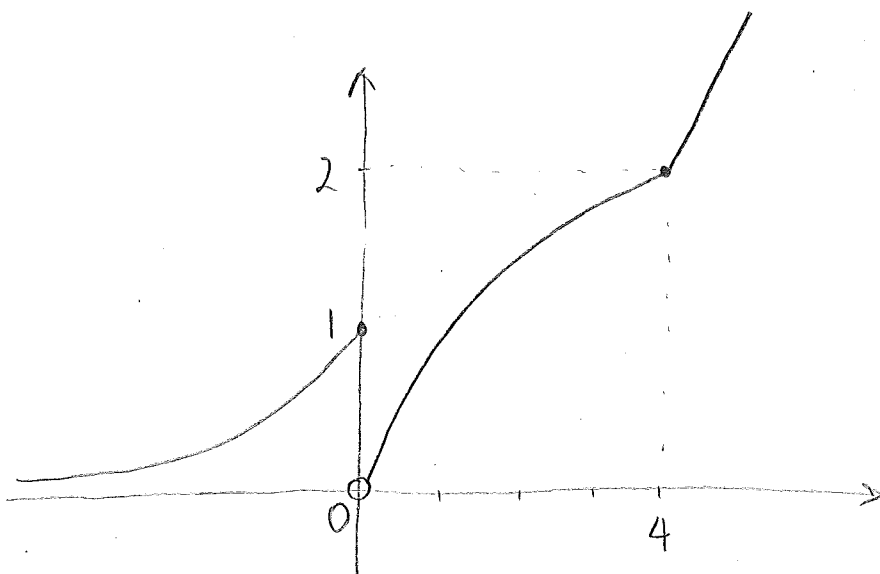
[True]

(c) The limit $\lim_{x \rightarrow 2} \frac{f(x)}{g(x)}$ does not exist if $g(2) = 0$. True / False

[False]

8 marks 3. (a) Sketch the graph of the following function,

$$f(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } 0 < x < 4 \\ 2x - 6 & \text{if } x \geq 4. \end{cases}$$



(b) Which one of the following statements is the best reason why $f(x)$ is **not continuous** at $x = 0$?

- (i) $f(0)$ does not exist.
- (ii) $\lim_{x \rightarrow 0} f(x)$ does not exist.
- (iii) Both (i) and (ii).
- (iv) $f(0)$ and $\lim_{x \rightarrow 0} f(x)$ exist, but they are not equal.

[ii]

(c) Which one of the following statements is the best reason why $f(x)$ is **continuous** at $x = 4$?

- (i) $f(4)$ exists.
- (ii) $\lim_{x \rightarrow 4} f(x)$ exists.
- (iii) Both (i) and (ii).
- (iv) $f(4)$ and $\lim_{x \rightarrow 4} f(x)$ exist, and they are equal.

[iv]

(d) Is $f(x)$ differentiable at $x = 4$? Explain why or why not.

No, $f(x)$ is not differentiable at $x=4$ because the curve has a corner at $x=4$

(e) Now consider the function

$$g(x) = \begin{cases} e^x & \text{if } x \leq 0 \\ \sqrt{x} & \text{if } 0 < x < 4 \\ mx + b & \text{if } x \geq 4. \end{cases}$$

where m and b are constants. Find values for m and b such that $g(x)$ is differentiable at $x = 4$.

$$\begin{cases} m = \frac{1}{4} \\ b = 1 \end{cases}$$

6 marks 4. The following limit

$$\lim_{h \rightarrow 0} \frac{\frac{1}{(2+h)^2} - \frac{1}{4}}{h}$$

equals the derivative $f'(a)$ for some function f and some point a .

(a) Give an example of one such function f and point a .

$$\left[\begin{array}{l} f(x) = \frac{1}{x^2} \\ a = 2 \end{array} \right]$$

(b) Evaluate the limit above (without using any differentiation rules).

$$\left[-\frac{1}{4} \right]$$

(c) Confirm your answer to part (b) by computing the derivative $f'(a)$ for the function f and point a you identified in part (a) using appropriate differentiation rules.

$$\left[f'(x) = -2x^{-3} \right]$$

- 6 marks 5. If a rock is thrown upward on the planet Mars with a velocity of 10 m/s, its height (in metres) above the ground level after t seconds is given by

$$h(t) = 10t - 2t^2.$$

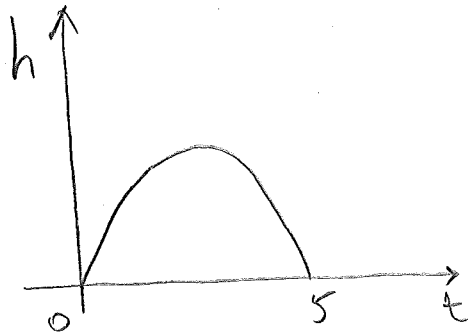
- (a) At what time t_h will the rock hit the surface?

$$[t_h = 5 \text{ s}]$$

- (b) Find the velocity of the rock when it hits the surface.

$$[v(5) = -10 \text{ m/s}]$$

- (c) Sketch a diagram of the rock's height as a function of time for $0 \leq t \leq t_h$.



8 marks

6. This question has three independent parts.

(a) Evaluate the following expressions:

$$\sin\left(\frac{2}{3}\pi\right) = \frac{\sqrt{3}}{2}$$

$$\cos\left(\frac{2}{3}\pi\right) = -\frac{1}{2}$$

(b) Find all points where the function $f(x) = x \cos x$ crosses the x -axis.

$$\left[\begin{array}{l} X = \frac{\pi}{2} + 2k\pi \\ \text{and} \\ X = \frac{3}{2}\pi + 2k\pi \end{array} \right]$$

(c) Find the equation of the tangent line to the curve $f(x) = x \cos x$ at the point $(\pi, f(\pi))$.

$$[y = -x]$$

7 marks

7. Differentiate the following functions. You may leave your answers unsimplified.

(a) $y = (1 + x^2)^{50}$.

$$\left[y' = 100x(1+x^2)^{49} \right]$$

(b) $f(x) = e^{\sqrt{3x}}$.

$$\left[f'(x) = \frac{3}{2\sqrt{3x}} e^{\sqrt{3x}} \right]$$

(c) $g(t) = \frac{\ln(3t-4)}{t}$.

$$\left[g'(t) = \frac{3t - (3t-4)\ln(3t-4)}{t^2(3t-4)} \right]$$

8 marks 8. Assume that the number of bacteria at time t (in minutes) follows an exponential growth model $P(t) = Ce^{kt}$, where C and k are constant. Suppose the count in the bacteria culture was 300 after 15 minutes and 1800 after 40 minutes. You may leave your answers unsimplified, i.e. in "calculator-ready" form.

(a) What was the initial size of the culture?

$$\left[C = \frac{300}{e^{\frac{3}{5} \ln 6}} \right]$$

(b) At what rate (in bacteria per minute) was the culture growing initially, i.e. at $t = 0$?

$$\left[P'(0) = \frac{300}{e^{\frac{3}{5} \ln 6}} \cdot \frac{\ln 6}{25} \right]$$

(c) When will the culture contain 3000 bacteria?

$$\left[\begin{aligned} t &= \frac{25}{\ln 6} \ln(10e^{\frac{3}{5} \ln 6}) \text{ min} \\ &= \frac{25 \ln 10}{\ln 6} + 15 \text{ min} \end{aligned} \right]$$

- 6 marks
9. When a pebble is thrown into a pond, circular ripples form on the water surface. Suppose $r(t)$ is the radius, in centimetres, of one such ripple at t minutes after the pebble falls into the water, and $A(r)$ is the surface area of a ripple of radius r .
- (a) Which of the following statements best explains the meaning of the composite function $A(r(t))$?
- (i) The radius of the ripple, in centimetres, at time t minutes.
 - (ii) The area of the ripple, in square centimetres, of radius r centimetres.
 - (iii) The area of the ripple, in square centimetres, at time t minutes.
 - (iv) The function f of the minutes and the area.
 - (v) The product of the area of the ripple, in square centimetres, by the ripple's radius at time t minutes.
- [iii]
- (b) Suppose the radius of the ripple increases at a rate of 20 centimetres per minute. How fast is the area of the ripple increasing when the radius is 10 cm? Include units. (Recall the area of a circle of radius r is $A = \pi r^2$.)

$$\left[\frac{dA}{dt} = 400\pi \frac{\text{cm}^2}{\text{min}} \right]$$

- 6 marks 10. Find equations of all lines having slope -1 that are tangent to the curve $y = \frac{1}{x-1}$.

$$\left[\begin{array}{l} y = -x + 3 \\ \text{and} \\ y = -x - 1 \end{array} \right]$$

- 4 marks 11. Show that there is at least one point on the curve $y = x^4 + 2x^2 - 4x - 1$ where the tangent line to the curve is horizontal.

$$\begin{aligned} y' &= 4x^3 + 4x - 4 \\ 4x^3 + 4x - 4 &= 0 \\ x^3 + x - 1 &= 0 \\ \text{apply IVT} \end{aligned}$$