MATH 110 April Exam, April 23rd, 2015
Duration: 150 minutes
*This test has **10 questions** on **13 pages**, for a total of 90 points.*

- Read all the questions carefully before starting to work.
- All questions require a full solution; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed**: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: _____ Last Name: _____

Student-No: _____ Section: _____

Signature: _____

Question:	1	2	3	4	5	6	7	8	9	10	Total
Points:	8	12	9	15	7	9	6	9	9	6	90
Score:											

Full-Solution Problems. In the following questions, justify your answers and **show all your work**. Unless otherwise indicated, **simplification of answers are required**.

8 marks 1. (a) Evaluate $\lim_{x\to-\infty} \dfrac{1-x-x^2}{2x^2-5}$.

(b) For what value of the constant c is the following function continuous everywhere

$$f(x) = \begin{cases} \frac{x-3}{x^2-9} & \text{if } x < 3 \\ cx^2 + \frac{3}{x} & \text{if } x \geq 3. \end{cases}$$

(c) Find the derivative of $g(x) = \ln\left(\dfrac{x+2}{x+3}\right)$.

(d) Find a function $f(x)$ whose derivative is $\sqrt[3]{x} + 3\sin x$.

12 marks 2. (a) The area A of a triangle with sides of length a and b enclosing an angle θ is

$$A = \frac{1}{2}ab\sin\theta.$$

Suppose the two sides a and b are both increasing at a rate of 2 cm/min, while the angle θ remains constant. Find an expression for the rate of change of the area of the triangle with respect to time in terms of a, b, and θ.

(b) Consider the function $f(t) = 4t - 4\cos t$ on the interval $[0, 2\pi]$. Find all critical numbers and the absolute maximum of f on the given interval.

(c) Find the equation of the tangent line to the curve $xy^3 + x^2y = 12$ at the point $(3, 1)$.

(d) Consider a function f that is continuous and increasing on $(-\infty, \infty)$. If $f(c) = 0$ for some number c, what can you say about the following limits of the function $h(x) = \dfrac{1}{f(x)}$:

$\lim_{x \to c^-} h(x) =$

$\lim_{x \to c^+} h(x) =$

Give an example of one such function $h(x)$ for $c = 1$.

9 marks 3. A curve has equation $y = f(x)$.

(a) Write down the definition of $f'(3)$ as a limit.

(b) Give a geometrical interpretation of the limit in part (a)

(c) Describe in your own words what it means to take the limit identified in part (a) and how it leads to the geometrical interpretation given in part (b).

(d) Suppose $f(x) = (3x-1)^2$. Find $f'(3)$ using the definition in part (a). No marks will be given to answers that do not involve a limit or calculations involving L'Hopital's rule.

15 marks

4. Let $f(x) = 3xe^{-(x^2/2)}$.

(a) State the domain of f.

(b) Find the y- and x-intercepts of f, if they exist.

(c) Find all the vertical and horizontal asymptotes of f, if they exist.

(d) Compute $f'(x)$.

(e) Find the intervals where f is increasing and the intervals where it is decreasing.

(f) Compute $f''(x)$.

(g) Find the intervals where f is concave up and the intervals where it is concave down.

(h) Make a large sketch of the graph of the function f. Make sure you identify the x-coordinate of any local extreme values and inflection points, if they exist.

7 marks

5. (a) Sketch the graph of a differentiable function f that satisfies ALL of the following conditions:

- $f > 0$ for all x.
- $f' > 0$ on $(-\infty, 0)$ and $f' < 0$ on $(0, \infty)$.
- $f'' > 0$ on $(-\infty, -1)$ and $(1, \infty)$ and $f'' < 0$ on $(-1, 1)$.

Make sure you label the axes and clearly identify the points on the graph that are related to the above conditions.

(b) In part (a) you sketched the graph of a function f. Now sketch the graph of its *derivative* f'. Make sure you identify the x-coordinate of any extreme values and inflection points (if they exist), you do not need to identify their y-coordinate.

9 marks

6. A cone shaped paper cup 8 cm wide (at the top) and 10 cm deep is full to the brim with water. A hole forms in the bottom of the cup, and the cup begins to lose water at a rate of 2 cm^3/min. How fast is the water level dropping when the water is 8 cm deep? (Recall that the volume of a cone of height h and base radius r is $\frac{1}{3}\pi r^2 h$.)

What would the answer be if the paper cup were shaped like a circular cylinder of height 10 cm and base diameter 8 cm and the water drained at the same rate as in the previous case (i.e. at 2 cm^3/min)? (Recall that the volume of a circular cylinder of height h and base radius r is $\pi r^2 h$.)

6 marks

7. A model used for the yield Y of an agricultural crop as a function of the nitrogen level N in the soil (measured in appropriate units) is

$$Y = \frac{5N}{k + N^2}$$

where k is a positive constant.

(a) What nitrogen level gives the best yield?

(b) For the nitrogen level found in part (a), what is the yield if $k = 4$?

9 marks

8. A small island is 2 km from the nearest point P on the straight shoreline of a large lake. If a woman on the island can row a boat 4 km per hour and can walk 5 km per hour, where should the boat be landed in order to arrive at a town 10 km down the shore from P in the least time? Show that your answer yields the absolute minimum.

9 marks 9. (a) Use an appropriate linear approximation to estimate $\sqrt{15}$.

(b) Is your answer in part (a) an overestimate, an underestimate, or exactly equal to, the actual value of $\sqrt{15}$? Justify your answer.

(c) Sketch a graph to illustrate your claim in part (b). Your sketch must show *clearly* the graph of the linear function used to compute the approximation in part (a) and how the approximation compares to the exact value of $\sqrt{15}$. Your sketch need not be in scale.

(d) Use the 2nd-degree Taylor Polynomial of $\sqrt{x}$ at 16 to approximate $\sqrt{15}$. You do not need to simplify your answer.

6 marks 10. Two runners start a race at the same moment and finish in a tie. Let f and g be the position functions of the two runners.

(a) Suppose it takes T minutes for the runners to complete the race. Consider the function $h(t) = f(t) - g(t)$. Find $h(T)$.

(b) Apply the Mean Value theorem to the function $h(t) = f(t) - g(t)$ and prove that at least one time during the race the runners have the same speed.

(c) If one runner overtakes the other one at least once during the race, show that there is at least one instant when the runners have the same acceleration.