WORKSHOP 2·9

Solutions

Warm-up

Below is a sketch of the bend on University Boulevard inside Pacific Spirit Park.

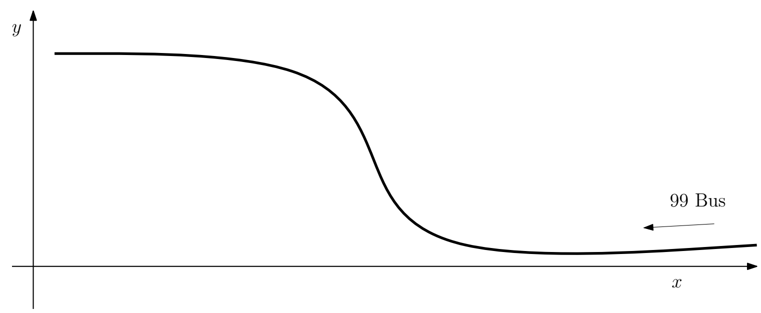
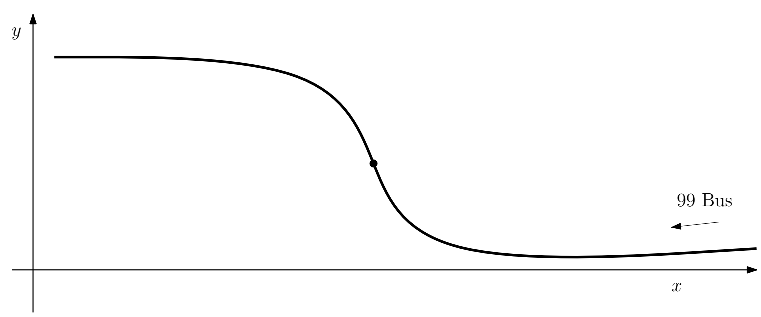


Figure 1: Plot of a trajectory around the infamous Wall of Campions.

Reproduce the sketch on the board as closely as possible. Notice that the curve can be described by a function $y = f(x)$. Find any inflection points of $f(x)$ on the graph and interpret what the 99 bus driver must be doing at that point in order to maintain the trajectory.

Solution: Below is the graph with a dot on the inflection point, where the concavity of the curve changes. An inflection point is where the concavity of a curve changes. At the inflection point on the



given graph, we see that the bus must change from steering towards its right to steering to the left, so at the inflection point the steering wheel should be centred as if the bus were moving in a straight line at that instant.

Main Problem

Researchers have modelled the thermic effect of food for a person using the function

$$F(t) = bte^{-t/c},$$

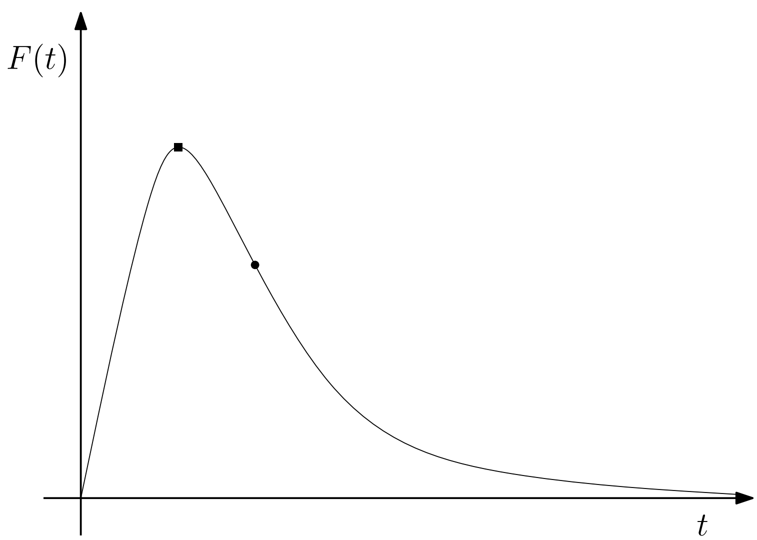
where $F(t)$ is the thermic effect of food (in kJ/hr *above the resting metabolic rate*), t is the number of hours that have elapsed since eating a meal, and b and c are positive constants.

Part (a):
Based on the information above, make a prediction of what the graph of $F(t)$ would look like. Then, using calculus verify your prediction; in particular, sketch the graph of $F(t)$, indicating any local extrema and inflection points. In this context, what interpretation would you give to $\lim_{t\to\infty} F(t)$? Does your y-intercept value make sense?

Part(b):
When is the thermic effect of food maximum? How would the maximum change if you changed the values of b and c? You may use numerical values and change b and c slightly to validate your results. From experimental data on metabolic rates, the researchers found that that for an average person, $b = 175.9$ and $c = 1.3$.

Solution:

Part (a):
A prediction: Since the metabolic rate tends to go on after a meal and eventually returns to a metabolic rest state, the thermic effect of food must be a positive function, initially increasing and reaching a maximum value, and then decreasing to eventually reach or approach a value equal to the initial value. Below is a rough sketch of the function according to our predictions.



Verify with calculus: We know that in the domain $t \geq 0$ the function $F(t)$ is continuous everywhere, so there can be no vertical asymptote. The only possible horizontal asymptote is when t goes to $+\infty$. We then compute the limit

$$\begin{aligned}\lim_{t\to\infty} F(t) &= \lim_{t\to\infty} bte^{-t/c}\\ &= b\lim_{t\to\infty} te^{-t/c}\\ &= 0.\end{aligned}$$

The last limit is zero as we have the quotient $\frac{t}{e^{t/c}}$ and the exponential will always dominate for large values of t. Therefore our horizontal asymptote is at $y = 0$, the x-axis.
In order to find any critical points we find the first derivative $F'(t)$:

$$F'(t) = be^{-t/c} - \frac{b}{c}te^{-t/c} = be^{-t/c}\left(1 - \frac{t}{c}\right).$$

To find the critical points we need to solve $F'(t) = 0$. Since b and $e^{-t/c}$ are both greater than zero, the only critical point is when

$$\begin{aligned} 1 - \frac{t}{c} &= 0 \\ t &= c. \end{aligned}$$

Our critical point, which we predict will be a local maximum is at $t = c$, noted by a square on the sketch. One can easily check that by using the first derivative test. When $0 < t < c$, $F' > 0$ (check it by verify the sign of F' at, say, $t = c/2$). If $t > c$, $F' < 0$ (verify it by using $t = 2c$ as test point). Thus F is increasing if $0 < t < c$ and decreasing if $t > c$ and $F(c) = (bc)e^{-1}$ is a local maximum.
Now, we need to determine concavity. We find the second derivative:

$$\begin{aligned} F''(t) &= \frac{d}{dt}\left(be^{-t/c}\left(1 - \frac{t}{c}\right)\right) \\ F''(t) &= -\frac{b}{c}e^{-t/c}\left(1 - \frac{t}{c}\right) - \frac{b}{c}e^{-t/c} \\ F''(t) &= -\frac{b}{c}e^{-t/c}\left(2 - \frac{t}{c}\right). \end{aligned}$$

To find any possible inflection points we need to solve

$$\begin{aligned} 2 - \frac{t}{c} &= 0 \\ t &= 2c. \end{aligned}$$

Furthermore, if $0 < t < 2c$, $F'' < 0$ (verify it by testing the sign of F'' at, say, $t = c$). If $t > 2c$, $F'' > 0$ (verify it by using $t = 3c$ as test point). Thus, F is concave down for $0 < t < 2c$ and concave up for $t > 2c$. At $t = 2c$ the concavity of the function changes, thus $(2c, F(2c))$ noted by a circle in the sketch, is an inflection point. Using this information, it turns out that our sketch prediction was about right. Figure 2 shows a numerically computed graph as a comparison.
Note that in this context the limit $\lim\limits_{t\to\infty} F(t)$ represents the increase of metabolic rate due to food ingestion a long time after a meal. It is reasonable to expect $\lim\limits_{t\to\infty} F(t) = F(0)$, because once the body has digested the food, the thermic effect of food must vanish, bringing the metabolic rate back to its resting value. Similarly, at $t = 0$ we just ate a meal, so it is predictable that the thermic effect will start from zero ($F(0) = 0$) and will quickly build up from that time to make the function continuous to just before we ate the food.

Part (b)
Based on the calculations above, the function $F(t)$ is always positive and has only one local maximum at $t = c$. Thus the thermic effect of food reaches its maximum value at $t = 1.3$ hours since the latest meal for an average person.
Finally, if c increases, the thermic effect of food reaches its maximum at a later time. In addition, if b decreases, the maximum thermic effect of food will be less intense. For the values of the parameters given in this problem, for an average person, $F(c) = \frac{bc}{e} = \frac{175.9\times 1.3}{e} \approx 84$ kJ/hr.

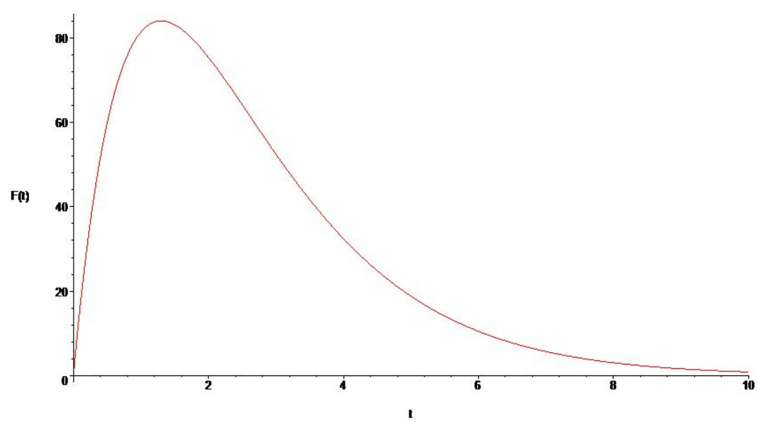


Figure 2: Numerically computed graph of metabolic rate.