WORKSHOP 2·6

Solutions

The following "reaction model" describes the reaction of pupils to light:

$$R = \frac{40 + 24x^{0.4}}{1 + 4x^{0.4}},$$

where R is the surface area (in mm^2) of the pupil, and x is the brightness (in W) of a light source directed at the pupil.

Part 1: Make predictions.
By thinking about our reaction to light, come up with predictions for what the graph of R ought to look like for the pupil diameter model. In particular:

(a) Should the graph be increasing or decreasing?

(b) Should the graph be positive or negative?

Be prepared to justify your predictions.
Next, come up with a plan to test your predictions. This part of the problem should be done without doing any calculations.

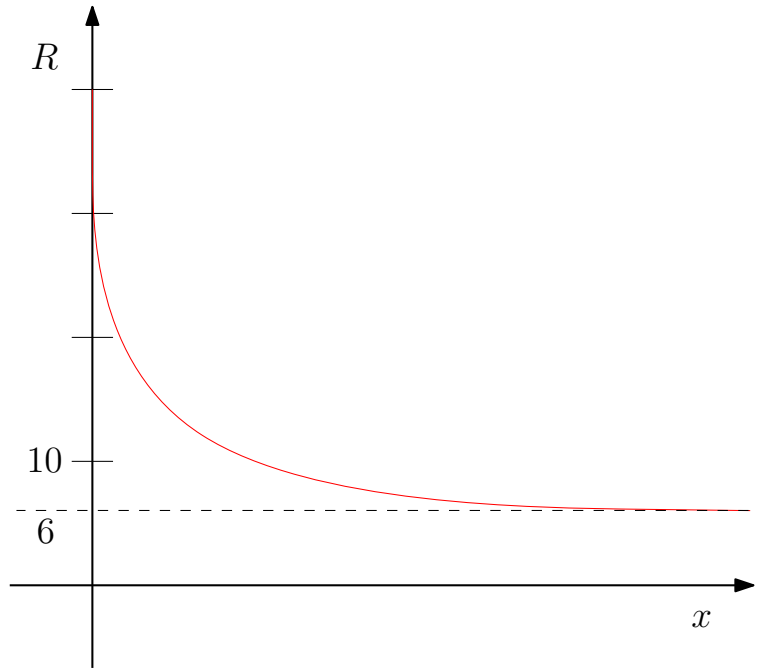
Solution:
From the experiment earlier, we saw that as the light exposure increases, the pupil shrinks in size, with its surface area being positive and within a range somewhere between zero and the surface of the iris circle.
As we increase the intensity of the light, the pupil may reach a surface area that cannot go any much further, so we expect the concavity of the function should be postive as it is decreasing and then flattens out.
In order to test the predictions, we shall compute the first derivative and check that its sign is always negative and the function is decreasing. To check the sign of the actual function. We may check small and large values of x, but we can figure it out easily using algebra and the concavity will be tested by finding the sign of the second derivative. If it is positive, the function will be concave up.

Part 2: Test your predictions.
After a TA has checked your predictions and your plan, carry out your plan and check your predictions by doing the appropriate calculations. After that, use your findings to make a rough sketch of R as a function of x.

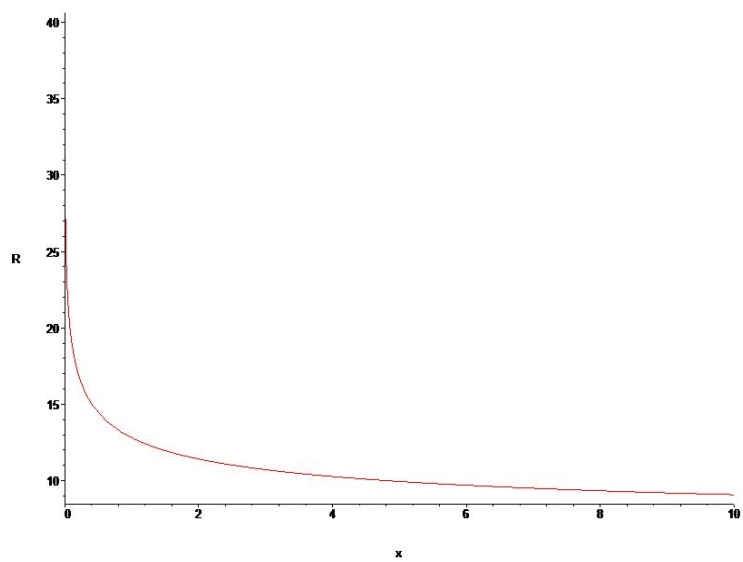
Solution:
First of all, we notice that x is non-negative, thus both the numerator and denominator of the function R are positive and R is also positive. It will be useful for the graph sketch to know that $R = 40$ when $x = 0$ and that if we input a very large number (or better still take the limit as x reaches infinity) the fraction approaches 6.
We can then use the quotient rule to find the first derivative:

$$\begin{aligned}
\frac{dR}{dx} &= \frac{\frac{d}{dx}(40 + 24x^{0.4}) \cdot (1 + 4x^{0.4}) - (40 + 24x^{0.4}) \cdot \frac{d}{dx}(1 + 4x^{0.4})}{(1 + 4x^{0.4})^2} \\
&= \frac{0.4 \cdot 24x^{-0.6} \cdot (1 + 4x^{0.4}) - (40 + 24x^{0.4}) \cdot 0.4 \cdot 4x^{-0.6}}{(1 + 4x^{0.4})^2} \\
&= \frac{9.6x^{-0.6} + 38.4x^{-0.2} - 64x^{-0.6} - 38.4x^{-0.2}}{(1 + 4x^{0.4})^2} \\
&= \frac{-54.4}{x^{0.6}(1 + 4x^{0.4})^2}.
\end{aligned}$$

Since x is non-negative, the fraction is negative because the numerator is negative and the denominator is positive. Thus the function is decreasing. Note that as x approaches 0, the derivatives approaches infinity so we will have a vertical tangent at $x = 0$.
Finally, we have a decreasing concave up function that goes from 40 at $x = 0$ to 6 as x grows larger. Moreover the tangent line is vertical at $x = 0$. The sketch should look like this:



Which has a similar shape to that actual plot, computed with the help of a computer:



Extra Problem

For the following two functions, identify all intervals where it is increasing or decreasing. Find and identify any local extrema.

(a) $y = x^4 e^{-x}$

(b) $g(t) = \frac{(t-4)^2}{\sqrt{6-t}}$

Solution

(a) The derivative of the function is

$$\begin{aligned} y' &= 4x^3 e^{-x} - x^4 e^{-x} \\ &= (4x^3 - x^4)e^{-x} \\ &= x^3(4-x)e^{-x}. \end{aligned}$$

We know that e^{-x} is always positive and x^3 has the same sign as x. $4 - x$ is positive for $x < 4$ and negative if $x > 4$. Multiplying the signs we get that the derivative is negative, therefore the function is decreasing for $x < 0$ and $x > 4$ and positive (function is increasing) if $0 < x < 4$. The extrema are located at $x = 0$ and $x = 4$. At $x = 0$ the function goes from decreasing to increasing, so it's a minimum. At $x = 4$ it goes from increasing to decreasing, so it's a maximum.

(b) The derivative of the function is

$$\begin{aligned} g'(t) &= \frac{2(t-4)\sqrt{6-t} - (t-4)^2 \frac{1}{2\sqrt{6-t}}(-1)}{6-t} \\ &= \frac{2(t-4)(6-t) + \frac{(t-4)^2}{2}}{(6-t)^{3/2}} \\ &= \frac{-2t^2 + 20t - 48 + 0.5t^2 - 4t + 8}{(6-t)^{3/2}} \\ &= \frac{-1.5t^2 + 16t - 40}{(6-t)^{3/2}} \end{aligned}$$

We know that $(6-t)^{3/2}$ is always positive in the domain $t < 6$ so we only need to find the sign of the numerator. Finding the roots using the quadratic formula for $a = -1.5$, $b = 16$ and $c = -40$ gives

$$\begin{aligned} \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} &= \frac{-16 \pm \sqrt{16^2 - 4(-1.5)(-40)}}{-3} \\ &= \frac{-16 \pm \sqrt{256 - 240}}{-3} \\ &= \frac{-16 \pm \sqrt{16}}{-3} \\ &= \frac{-16 \pm 4}{-3} \\ &= 4 \text{ or } 20/3. \end{aligned}$$

Therefore the only local extrema is at $x = 4$ because the other value is outside the domain. Moreover we know that if $x < 4$ the numerator is negative because a is negative and we are left of either roots and if $4 < x < 6$ the derivative is positive. Finally we have a decreasing function for $x \in (-\infty, 4)$, a local minimum at $x = 4$ and an increasing function for $x \in (4, 6)$.