

## WORKSHOP 2.5

### Solutions

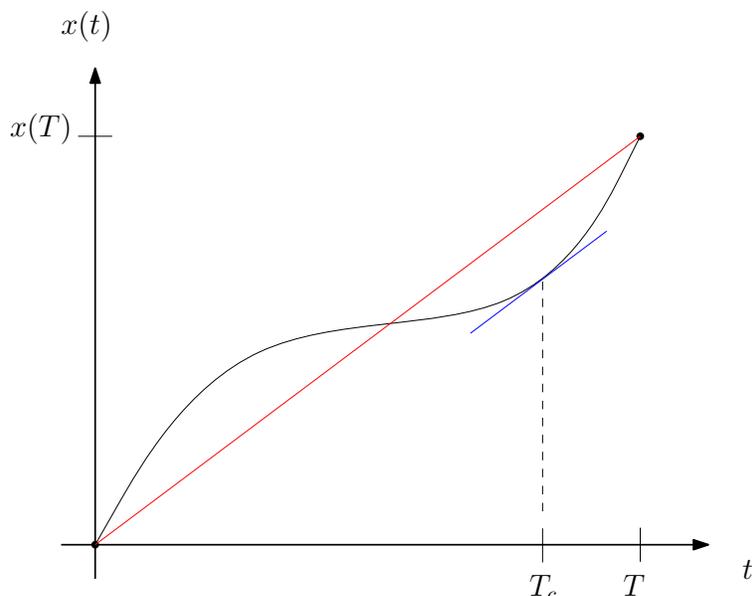
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#### Warm-up Problem

Question: You are a traffic court judge. A speeding ticket has been given on a road where the speed limit is 90 km/hr and you need to decide if the ticket is appropriate. The toll taker says, "Your Honor, based on the elapsed time from when the car entered the toll road until the car stopped at my booth, I know the average speed of the car was 103 kilometres per hour. I did not actually see the car speeding, but I know it was and I gave the driver a speeding ticket."

#### Solution:

Let's start by plotting the position of the car along the road against time:



Suppose the car's itinerary starts at  $t = 0$  and ends at  $t = T$  and let  $x(t)$  be the position of the car along the road at time  $t$ . We know  $x(t)$  is continuous and differentiable on the interval  $[0, T]$  as at any point in that interval the car has some instantaneous velocity.  $x(t)$  satisfies the assumptions to the Mean Value Theorem, therefore there must be a time  $T_c$  in the interval  $(0, T)$  when the derivative has the same value as the slope of the secant line from 0 and  $T$  (in red in the figure).

$$x'(T_c) = \frac{x(T) - x(0)}{T_c - 0} = \frac{x(T)}{T}.$$

However, the slope of the secant line is also the average velocity and its value is of 103 km/h. Therefore, at  $T_c$ , the instantaneous velocity, given by  $x'(T_c)$  is also of 103 km/h, which is higher than the limit, so the ticket would be justified.

## Main Problem

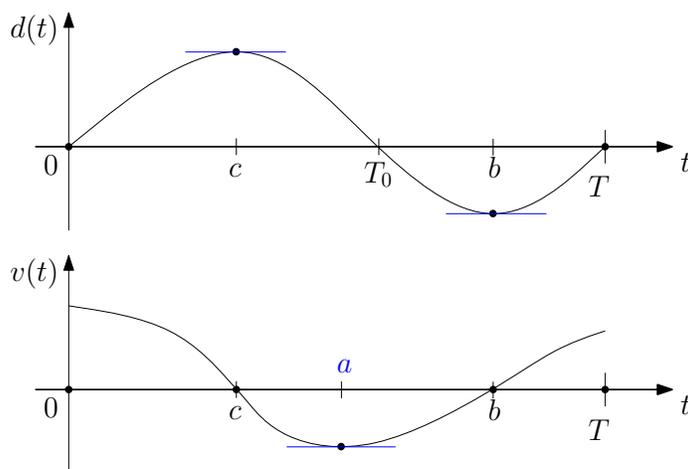
- (a) Two runners start a race at the same moment and finish in a tie. Prove that at least one time during the race the runners have the same speed.
- (b) If one runner overtakes the other one at least once during the race, show that there is at least one instant when the runners have the same acceleration.

### Solution:

(a) Let  $x_1(t)$  and  $x_2(t)$  be the positions of each runner at time  $t$  and let the race start at  $t = 0$  and end at  $t = T$  and the total distance be  $L$ . Both functions are continuous and differentiable on the interval  $[0, T]$ , therefore we could use the Mean Value Theorem on each function, but that method will not work.

The reason it will not work is that if you use the theorem on each function separately, you will find two separate times  $t_1$  and  $t_2$ , when each runner reaches the average speed  $L/T$ . Generally those two times will not coincide and this particular method will fail, because we are applying the theorem on two different functions.

If we only consider the distance between the two runners, given by  $d(t) = x_2(t) - x_1(t)$ , we get a function that is also continuous and differentiable on  $[0, T]$  if we plot the following function we should get a function like the top of the following figure. We have  $d(0) = d(T) = 0$  so we may use Rolle's Theorem to find that



there exists a point  $c$  between 0 and  $T$  where  $d'(c) = 0$ . If we develop the derivative of  $d(t)$ , we obtain

$$\begin{aligned}d'(t) &= (x_2(t) - x_1(t))' \\d'(t) &= x_2'(t) - x_1'(t).\end{aligned}$$

Therefore we have that  $x_2'(c) - x_1'(c) = 0$ , so at time  $c$ ,

$$x_1'(c) = x_2'(c)$$

(b) In this part, we know that there is a time  $T_0$  when one runner overtakes the other. Thus we also have that  $d(T_0) = 0$ . We can then use the Mean Value Theorem on  $d(t)$  on the intervals  $[0, T_0]$  and  $[T_0, T]$  to find two different times  $c$  and  $b$  when  $d'(c) = d'(b) = 0$ .

Let  $v(t) = d'(t)$ , then  $v(t)$  is also continuous and differentiable on  $[0, T]$  as each runner has an acceleration at any given time. An example of a graph of  $v(t)$  is given in the lower plot of the last figure. Moreover we know that  $v(c) = v(b) = 0$ , so we may use Rolle's Theorem on the interval  $[c, b]$  to find a time  $a$  in the interval when  $v'(a) = 0$ , from the definition of  $d(t)$  we have that  $v'(t) = d''(t) = x_2''(t) - x_1''(t)$ , so we obtain at time  $a$ :

$$x_1''(a) = x_2''(a),$$

or the two runners have the same acceleration at  $t = a$ .