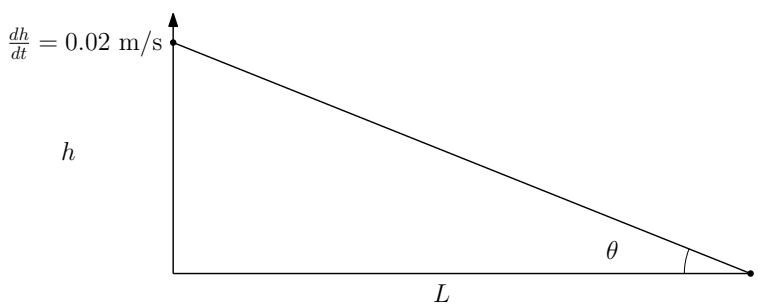
WORKSHOP 2·4

Solutions

Warm-up

> Question: Suppose you are staring at an equation written on the chalkboard when your instructor pushes up the chalkboard to access the board underneath. You keep staring at the equation as the board moves up at a rate of 2 cm/s. To keep your eyes focused on the equation, at what rate must you rotate them when the board is 1 m higher than its initial position?

Solution: We begin with a diagram. Let's assume the equation starts at the student's eye level and that he is seated at a distance L from the board.



We need to find the rate of change of the angle θ with time when the equation has risen by 1 meter and we are given the rate of change of the height of the equation:

$$\frac{dh}{dt} = 0.02 \text{ m/s}.$$

We can use the formula of the tangent to relate the angle and the height:

$$\tan\theta(t) = \frac{h(t)}{L}.$$

We noted here what quantities depend on time by putting the variable next to h. If we differentiate, we get the following:

$$\begin{aligned}\frac{1}{\cos^2\theta(t)} \cdot \frac{d\theta(t)}{dt} &= \frac{1}{L} \cdot \frac{dh(t)}{dt} \\ \frac{d\theta}{dt} &= \frac{\cos^2\theta}{L} \cdot \frac{dh}{dt}.\end{aligned}$$

We may use the ratios of the right triangle to compute the cosine function:

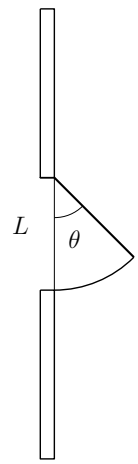
$$\cos\theta = \frac{L}{\sqrt{L^2 + h^2}}.$$

Substituting back in the angle derivative equation when $h = 1$ m we finally obtain

$$\begin{aligned}\frac{d\theta}{dt} &= \frac{\cos^2(\theta)}{L} \cdot \frac{dh}{dt} \\ \frac{d\theta}{dt} &= \frac{L}{L^2 + 1 \text{ m}^2} \cdot 0.02 \text{ m/s},\end{aligned}$$

Main Problems

1. When a door swings open, the door sweeps out a pie-shaped area of the floor. Suppose the door to this classroom swings open at 2 radians per second. Determine how quickly the area being swept out is increasing.



We are given the rate of change of the angle of the door with the wall:

$$\frac{d\theta}{dt} = 2 \text{ radians/s}.$$

We wish to know the rate of change of the swept area $\frac{dA}{dt}$.

To relate these two quantities we wish to know the formula of the swept area. The shape is a fraction of $\frac{\theta}{2\pi}$ of a circle, so its surface area must be:

$$A = \frac{\theta}{2\pi} \cdot \pi L^2.$$

Only A and θ depend on time here. When taking the derivative, we obtain

$$\frac{dA}{dt} = \frac{d\theta}{dt} \cdot \frac{L^2}{2}$$
$$\frac{dA}{dt} = 2 \text{ s}^{-1} \cdot \frac{L^2}{2}.$$

In the end L has distance dimensions, so we get a rate of change of area over time in units of distance squared over units of time.

2. Suppose a student is walking from one corner of the classroom to the opposite corner along the diagonal at a rate of 1.6 m/s. How fast is the distance from the student to one of the other corners of the room (not the one he is heading to or left from) changing when he is half way to the opposite corner?

Solution: We begin with a diagram:

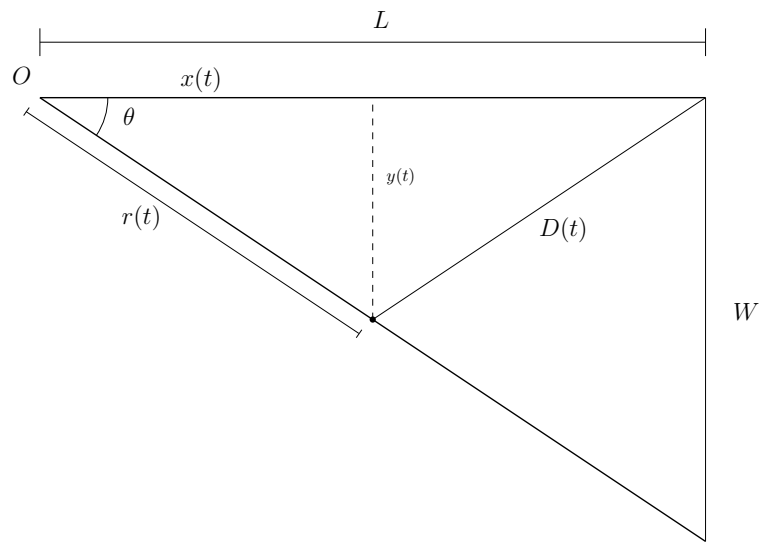
We are given the rate of change of the distance $r(t)$:

$$\frac{dr}{dt} = 1.6 \text{ m/s}.$$

We are looking for the rate of change of the distance $D(t)$ when we are halfway from the other corner.

We can find an expression for quantities $x(t)$ and $y(t)$ using trigonometric functions and we obtain:

$$x(t) = r(t)\cos\theta,$$
$$y(t) = r(t)\sin\theta.$$



Then we may use the distance formula to find a relation between the distances $r(t)$ and $D(t)$:

$$D(t)^2 = (L - x(t))^2 + (y(t))^2.$$

Implicit differentiation gives us

$$\begin{aligned} 2\frac{dD}{dt}D &= -2\frac{dx}{dt}(L-x) + 2\frac{dy}{dt}y \\ \frac{dD}{dt} &= \frac{-\frac{dx}{dt}(L-x) + \frac{dy}{dt}y}{D}. \end{aligned}$$

Let t_c be the time when the student is in the centre of his trajectory. We have at that time:

$$\begin{aligned} x(t_c) &= \frac{L}{2}, \\ y(t_c) &= \frac{W}{2}, \\ \frac{dx(t_c)}{dt} &= \frac{dr(t)}{dt}\cos\theta, \\ \frac{dy(t_c)}{dt} &= \frac{dr(t)}{dt}\sin\theta, \\ D(t_c) &= \sqrt{(L/2)^2 + (W/2)^2}, \end{aligned}$$

with

$$\begin{aligned} \cos\theta &= \frac{L/2}{r(t_c)} = \frac{L/2}{\sqrt{(L/2)^2 + (W/2)^2}} \\ \sin\theta &= \frac{W/2}{r(t_c)} = \frac{W/2}{\sqrt{(L/2)^2 + (W/2)^2}} \end{aligned}$$

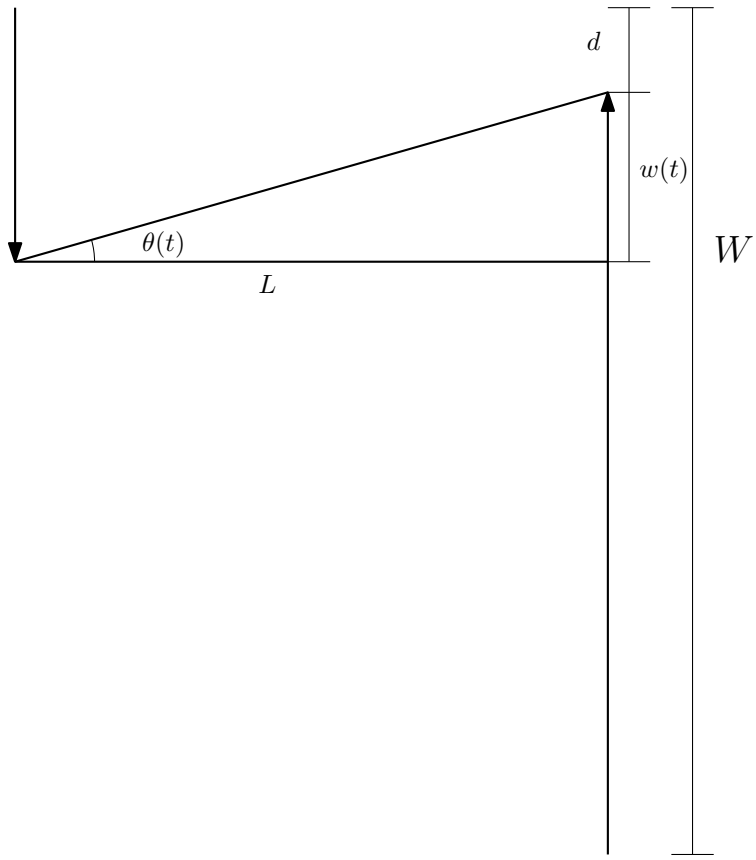
Substituting all of this gives us, after simplification,

$$\begin{aligned} \frac{dD}{dt} &= \frac{- \cdot 1.6 \text{ m/s} \cdot L^2 + \cdot 1.6 \text{ m/s} \cdot W^2}{4((L/2)^2 + (W/2)^2)} \\ \frac{dD}{dt} &= \frac{1.6 \text{ m/s} \cdot (W^2 - L^2)}{(W^2 + L^2)}. \end{aligned}$$

3. Suppose two students standing at opposite corners of the room start walking along the wall in opposite directions. One of them walks at a constant speed of $\frac{3}{4}$ metres per second. At the same time, the other student walks at a constant velocity of $\frac{1}{4}$ meters per second while recording his friend's movements on his phone.

   How fast must he rotate his phone in order to keep it focused on his friend when his friend is 1 metre from the opposite wall?

   Solution: We begin with a diagram:



   We wish to know the rate of change of the angle $\theta'(t)$ at a time when the right student is one metre from the opposite wall.

   We know that one student walks at 3/4 meters per second and the other at 1/4 meters per second in the opposite direction. This way the distance between them along the parallel walls $w(t)$ has the following rate of change:

$$\frac{dw(t)}{dt} = 1 \text{ m/s}.$$

   We can then use the tangent function to find a relationship between w and θ:

$$\tan\theta(t) = \frac{w(t)}{L}.$$

   We then use implicit differentiation:

$$\frac{\frac{d\theta(t)}{dt}}{\cos^2\theta(t)} = \frac{1}{L}\cdot\frac{dw(t)}{dt}.$$

Since t appears in the formulation of velocity we need to compute the time when we are one metre from the end.

The distance $w(t)$ may be formulated as

$$w(t) = 1 \text{ m/s} \cdot t - W.$$

We chose this formulation because on our diagram $h(t)$ initially points downwards and would be negative and we wish to have a positive distance at the end.

At time t_c, the right student is one metre from the opposite wall ($d = 1$):

$$\frac{3}{4} \text{ m/s} \cdot t_c = W - 1 \text{ m}.$$

We can then solve to get:

$$t_c = \frac{4}{3} \text{ s/m}(W - 1 \text{ m}).$$

We may use the tangent function to find $\cos\theta(t_c)$:

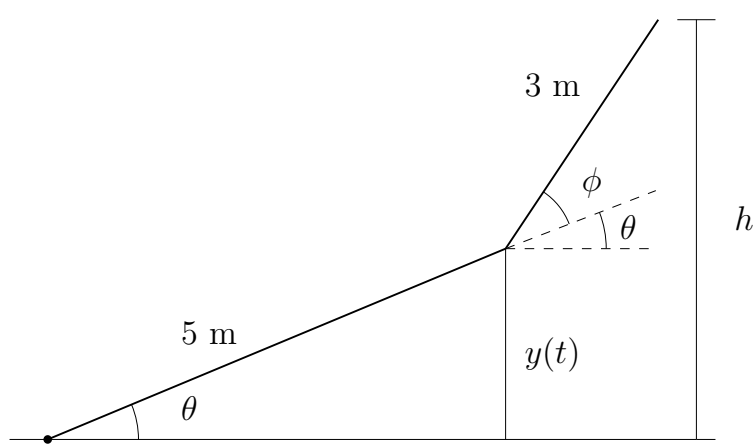
$$\cos\theta(t_c) = \frac{L}{\sqrt{L^2 + w(t_c)^2}}.$$

We then substitute in the rate of change formula to get

$$\begin{aligned}
\frac{d\theta}{dt} &= \frac{\cos^2\theta}{L} \cdot \frac{dw}{dt} \\
&= \frac{L}{L^2 + w(t_c)^2} \cdot 1 \text{ m/s} \\
&= \frac{L}{L^2 + \left(\frac{1}{3}W - \frac{4}{3} \text{ m}\right)^2} \cdot 1 \text{ m/s}.
\end{aligned}$$

Extra Problem

A city bus company created a new type of access ramp, composed of a first ramp of 5 metres bolted to another one of 3 metres. When pulling up the ramp, it also folds itself in two, as in the following diagram:



Two motors control the rate of change of the angles θ and ϕ to $\pi/6$ per second each (in radians). Find the rate of change of the height h of the tip of the second ramp after 1 second.

Solution: First of all notice the extra details on the diagram.
Let $y(t)$ be the height of the tip of the first ramp from the floor, we know that

$$y(t) = 5 \text{ m} \cdot \sin\theta(t).$$

We can then use parallel lines to find that the height of the tip of the second ramp is:

$$h(t) = y(t) + 3 \text{ m} \cdot \sin(\theta(t) + \phi(t)) = 5 \text{ m} \cdot \sin\theta(t) + 3 \text{ m} \cdot \sin(\theta(t) + \phi(t))$$

Taking the derivative, we get

$$h'(t) = 5 \text{ m} \cdot \cos\theta(t) \cdot \frac{d\theta(t)}{dt} + 3 \text{ m} \cdot \cos(\theta(t) + \phi(t)) \cdot (\frac{d\theta(t)}{dt} + \frac{d\phi(t)}{dt}).$$

After one second we know that $\theta(1) = \phi(1) = \frac{\pi}{6}$, we also know from the problem statement that

$$\frac{d\theta(t)}{dt} = \frac{d\phi(t)}{dt} = \frac{\pi}{6} \text{ s}^{-1}.$$

Substituting into the rate of change equation, we get

$$\begin{aligned} h'(1) &= 5 \text{ m} \cdot \cos\frac{\pi}{6} \cdot \frac{\pi}{6} \text{ s}^{-1} + 3 \text{ m} \cdot \cos\left(\frac{\pi}{6} + \frac{\pi}{6}\right) \cdot \left(\frac{\pi}{6} \text{ s}^{-1} + \frac{\pi}{6} \text{ s}^{-1}\right) \\ &= 5 \text{ m} \cdot \cos\frac{\pi}{6} \cdot \frac{\pi}{6} \text{ s}^{-1} + 3 \text{ m} \cdot \cos\left(\frac{\pi}{3}\right) \cdot \left(\frac{\pi}{3} \text{ s}^{-1}\right) \\ &= 5 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\pi}{6} \text{ m/s} + 3 \cdot \frac{1}{2} \cdot \frac{\pi}{3} \text{ m/s} = \frac{5\sqrt{3} + 6}{12}\pi \text{ m/s} \end{aligned}$$