WORKSHOP 2·13

Solutions (B)

Problem 3

> **Let a and b be constants. Show that if the curves $xy = a$ and $x^2 - y^2 = b$ intersect each other, they do so at right angles.**

Solution:
Recall that a two functions f and g intersect at a right angle if and only if we have $f'(x_0) = -1/g'(x_0)$ at the intersection input value x_0.
Let (x_0, y_0) be an intersection of the two curves. By implicit differentiation we find the following equation for each curve

$$\begin{aligned} y + xy'_a &= 0 \\ 2x - 2yy'_b &= 0, \end{aligned}$$

where the a subscript corresponds to the $xy = a$ curve and the b subscript correspond to the other curve. Solving for y' in each equation we obtain

$$\begin{aligned} y'_a &= \frac{-y_0}{x_0} \\ y'_b &= \frac{x_0}{y_0}. \end{aligned}$$

We then obtain that $y'_a = -1/y'_b$ and the curves must therefore intersect at a right angle.

Problem 4

> **Sketch the graph of $f(x) = \frac{\sin x}{e^x}$ on the interval $[0, \pi]$.**

Solution:
First we need to find the domain of the function. We know that both $\sin x$ and e^x are defined for any real number x and that e^x is never zero, so the domain of $f(x)$ would be all the real numbers, but in the context of the problem statement, we only need to look between 0 and π, o our domain will be the interval $[0, \pi]$.
Then we need to find the y-intercept and any roots of $f(x)$. We obtain $f(0) = 0$ so the origin is a root and the y-intercept. Any other root is located wherever $\sin x = 0$ and this only happens at integer multiples of π so our only other root in the domain is $x = \pi$.
There are no asymptotes as our function is defined for all real numbers (no vertical asymptotes) and our domain does not allow us to go to infinity on either direction on the x-axis (no horizontal asymptote).
The next step is to find any critical points, so we need to compute the first derivative

$$f'(x) = \frac{e^x \cos x - e^x \sin x}{(e^x)^2} = \frac{\cos x - \sin x}{e^x}.$$

Setting this equal to zero, we find that the critical points are located where $\sin x = \cos x$. This happens when $x = \pi/4 + n\pi$ for any integer n, so the only possible critical point in our interval is $x = \pi/4$.
To identify the critical point we compute the second derivative

$$f''(x) = \frac{(-\sin x - \cos x)e^x - (\cos x - \sin x)e^x}{(e^x)^2} = \frac{-2\cos x}{e^x}$$

$$f''\left(\frac{\pi}{4}\right) = \frac{-sqrt(2)}{e^{\pi/4}} < 0.$$

Since the second derivative is negative at the critical point, it will be a local maximum (the value of the function will be $\frac{\sqrt{2}}{e^{\pi/4}}$ at the local minimum). Moreover we find an inflection point at $x = \frac{\pi}{2}$ because the second derivative is zero there.

Then we may build a big sign chart for the values of the function and its derivatives in between the critical point, the inflection point and the endpoints

x	0		$\pi/4$		$\pi/2$		π
$f(x)$	0	$+$	$+$	$+$	$+$	$+$	0
$f'(x)$	$+$	$+$	0	$-$	$-$	$-$	$-$
$f''(x)$	$-$	$-$	$-$	$-$	0	$+$	$+$

Following the information about the roots, critical points and inflection point we may sketch the graph of the function and it should look like the following