WORKSHOP 2·11

Solution

Question:
Cut out four equal squares from the four corners of your piece of paper, and fold up the sides to form a box of maximal volume. You must justify your solution on the board.

You may use 12” of masking tape to construct your box.

Solution:
The box should be cut and folded according to figure 1.

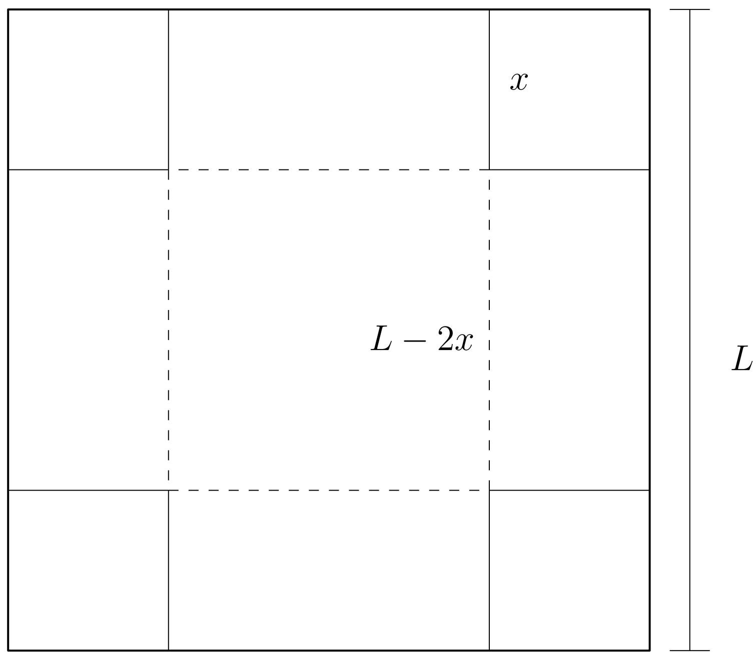

Figure 1: Box cutting and folding diagram.

Let’s first compute the volume of the box. If the full length of the square is L and we cut square corners of length x the base of the box will be a square of length $L - 2x$ and its height will be x. The volume is thus:

$$V(x) = x(L - 2x)^2.$$

Note that the volume only depends on the length of the sides of the cutted corner pieces x.
To find its maximal value we need to find the critical points, so we find the first derivative:

$$V'(x) = (L - 2x)^2 + 2x(L - 2x) \cdot (-2) = (L - 2x)(L - 6x).$$

Setting the derivative equal to zero we have two solutions:

$$x = L/2 \text{ or } L/6.$$

Note that if $x = L/2$ then the volume will be zero, so that point is probably not a maximum, so let’s look at $x = L/6$. Taking the second derivative we obtain

$$V''(x) = -2(L - 6x) - 6(L - 2x) = -8L + 24x.$$

Substituting $x = L/6$ we get

$$V''(L/6) = -8L + 24(L/6) = -8L + 4L = -4L < 0,$$

So the function is concave down and we have a local maximum.
We also have constraints on our values of x, as we may only use up to 12 inches of tape. To build our box, we need $4x$ of tape to join the walls together, so we need $4x \leq 12$, and we know that $x \geq 0$. So, $0 \leq x \leq 3$. We need to check volume values for $x = 0$, $x = 3$, and $x = L/6$ to determine the global maximum. We will now explicitly use the fact that $L = 8$ inches for the paper. At $x = 0$ the volume will be 0. At $x = 3$ we have

$$V(3) = (3)(8-6)^2 = 12.$$

At $x = L/6 = 8/6 = 4/3$ we have

$$V(4/3) = (4/3)(8 - 4/3)^2 = 4 \cdot 20^2/27 \approx 59.$$

The volume is highest at our critical point, so our box will have a maximal volume of $4 \cdot 20^2/27$ cubic inches at a height of $x = 4/3$ inches.