Warm-up Problem

Sketch the graph of a function that satisfies the following conditions:

1. the domain of \( f \) is all \( x \neq -1 \).
2. \( f(0) = f(4) = 0 \).
3. \( \lim_{x \to -1} f(x) = \infty \).
4. \( f'(0) = f'(3) = 0, f' < 0 \) if \( x < 3 \) and \( f' > 0 \) if \( x < -1 \) and \( x > 3 \).
5. \( f'' < 0 \) if \( 0 < x < 2 \) and \( f'' > 0 \) if \( x < -1, -1 < x < 0 \) and \( x > 2 \).

Solution:
There are multiple ways to build such a function. We suggest to build a chart noting any roots, critical points and inflection points and find where the function is positive / negative, increasing / decreasing, concave up / concave down.

1. Identification of points.
   Two roots are initially given at \( x = 0 \) and \( x = 4 \). There is a critical point at \( x = -1 \) as the function is not defined there. We have two more critical points at \( x = 0 \) and \( x = 3 \). The inflection points are at \( x = 0 \) and \( x = 2 \) as the second derivative changes sign at those values.

2. Table
   In order to find the sign of the function we look around the roots. We know the function is decreasing before and after \( x = 0 \) so the function is positive for \( x < 0 \) and negative for \( x > 0 \). At \( x = 4 \), the function is increasing so it will be negative for \( x < 4 \) and positive for \( x > 4 \). To get the sign of the function for \( x < -1 \), we know from condition 3 that the function is positive close to \(-1\). There might be a root for \( x < -1 \), but it is not specified.

<table>
<thead>
<tr>
<th>x</th>
<th>-1</th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>( f(x) )</td>
<td>(+)</td>
<td>DNE</td>
<td>+</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>( f'(x) )</td>
<td>+</td>
<td>DNE</td>
<td>-</td>
<td>0</td>
<td>-</td>
</tr>
<tr>
<td>( f''(x) )</td>
<td>+</td>
<td>DNE</td>
<td>+</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

3. Graph sketching
   The last step is to sketch the graph interval by interval on the sign table. It is important to get the vertical asymptote at \( x = -1 \) from condition 3 and for the function to be flat at \( x = 0 \) from condition 4.

Note that there are many ways to sketch the function. For example for \( x < -1 \) we could have our function decreasing below \( y = 0 \) and keep increasing and being concave up.
Main Problem

Now that we know how to sketch functions, it may be prudent to return to the ongoing zombie apocalypse. As you may recall from last term, we are in the midst of a zombie pandemic. Scientists have discovered that the fraction of the population that has been infected by the zombie virus can be determined in a precise way as a function of time. They have released their findings to the public, which state that the fraction \( f(t) \) of the population that has been infected \( t \) years after the initial outburst is given by the following function:

\[
f(t) = \frac{1}{1 + 9e^{-t}}.
\]

Based on your knowledge of zombie infection, predict the shape of the graph of the function \( f(t) \), and draw your prediction on the board. Use calculus to sketch the actual graph of the function \( f(t) \). At what time does the rate of infection reach its maximum value?

Solution:

1. Prediction
   People do not recover from the virus. Thus, \( f(t) \) should be an increasing function of \( t \), and \( f(t) \) approaches the total fraction of the population, with a value of 1, as \( t \to \infty \)

2. Domain
   \( f(t) \) is defined for \( t \geq 0 \). Thus, the domain of \( f(t) \) is \( t \geq 0 \).

3. Intercepts
   At \( t = 0 \), \( f(t) = 1/10 \). \( f(t) \neq 0 \) for any \( t \geq 0 \), so we don’t have any \( y \)-intercepts.

4. Asymptotes
   \( \lim_{t \to \infty} 1 = 1 \). \( \lim_{t \to \infty} 1 + 9e^{-t} = 1 \). Thus,

   \[
   \lim_{t \to \infty} \frac{1}{1 + 9e^{-t}} = \frac{1}{1} = 1.
   \]
5. Critical Points and intervals of increase/decrease

\[ f'(t) = \frac{9e^{-t}}{(1 + 9e^{-t})^2} \]

\(9e^{-t} > 0\) for all \(t \geq 0\). Thus, \(f'(t) > 0\) for all \(t \geq 0\). So, \(f(t)\) is an increasing function with no critical points.

6. Inflection points and concavity

From the quotient rule and some algebra,

\[ f''(t) = \frac{9e^{-t}(9e^{-t} - 1)}{(1 + 9e^{-t})^3}. \]

\(e^{-t} > 0\) for \(t \geq 0\). Thus, \(f''(t) = 0\) if \(9e^{-t} - 1 = 0\), which (after some algebra) occurs if \(t = \ln(9) \approx 2.2\). \(9e^{-t} - 1 > 0\) if \(t < \ln(9)\) and \(9e^{-t} - 1 < 0\) if \(t > \ln(9)\). Thus, \(f''(t) > 0\) if \(t < \ln(9)\) and \(f''(t) < 0\) if \(t > \ln(9)\). Therefore, \(f(t)\) is concave-up for \(0 \leq t < \ln(9)\), \(f(t)\) is concave-down for \(t > \ln(9)\), and \(f(t)\) has an inflection point at \(t = \ln(9)\).

The sketch of \(f(t)\) should look somewhat similar to the graph of \(f(t)\):

![Graph of f(t)](image)

Figure 2: graph of \(f(t)\).

The rate of infection reaches its maximum value when \(f'(t)\) is maximum, which occurs at the inflection point of \(f(t)\), at \(t = \ln(9)\).