WORKSHOP 2·2

Solution

Question:
Cut out four equal squares from the four corners of your piece of paper, and fold up the sides to form a box of maximal volume. You must justify your solution on the board.

You may use 12” of masking tape to construct your box.

Solution:
The box should be cut and folded according to figure 1.

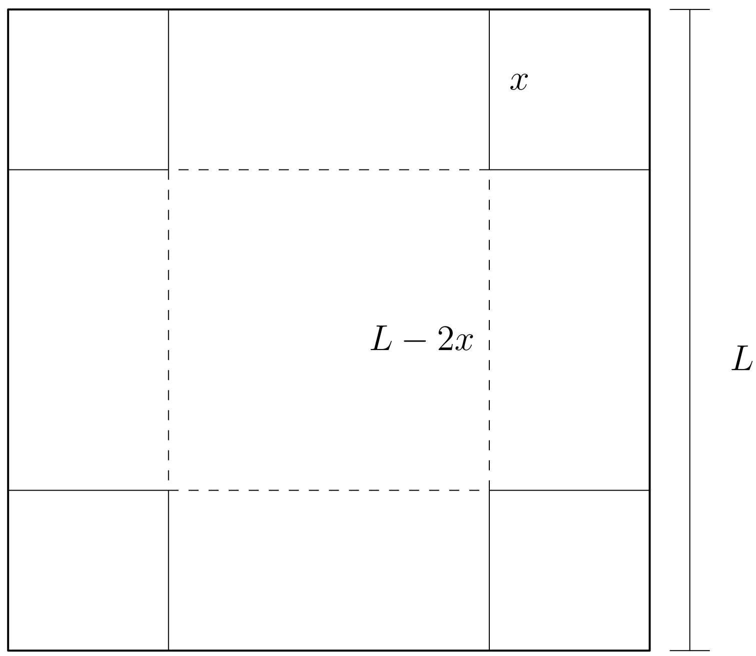

Figure 1: Box cutting and folding diagram.

Let’s first compute the volume of the box. If the full length of the square is L and we cut square corners of length x the base of the box will be a square of length $L-2x$ and its height will be x. The volume is thus:

$$V(x) = x(L-2x)^2.$$

Note that the volume only depends on the length of the sides of the cutted corner pieces x.
To find its maximal value we need to find the critical points, so we find the first derivative:

$$V'(x) = (L-2x)^2 + 2x(L-2x)\cdot(-2) = (L-2x)(L-6x).$$

Setting the derivative equal to zero we have two solutions:

$$x = L/2 \text{ or } L/6.$$

Note that if $x = L/2$ then the volume will be zero, so that point is probably not a maximum, so let’s look at $x = L/6$. Taking the second derivative we obtain

$$V''(x) = -2(L-6x) - 6(L-2x) = -8L + 24x.$$

Substituting $x = L/6$ we get

$$V''(L/6) = -8L + 24(L/6) = -8L + 4L = -4L < 0,$$

So the function is concave down and we have a local maximum. Let's compute its value:

$$V(L/6) = (L/6)(L - L/3)^2 = (L/6)(4L^2/9) = 2L^3/27.$$

We also have constraints to our values of x, as we may only use up to 12 inches of tape. To build our box we need $4x$ of tape to join the walls together, so we need $0 \leq x \leq 3L/8$ if our initial square is 8 inches long. We then need to check volume values for $x = 0$ and at $x = 3L/8$ and the highest volume will be the global maximum.
At $x = 0$ the volume will be 0. At $x = 3L/8$ we have

$$V(3L/8) = (3L/8)(L/4)^2 = 3L^3/128 < 2L^3/27$$

The volume is highest at our critical point than at the value at the bounds, so our box will have a maximal volume of $2L^3/27$ for a box with a height of $x = L/6$.