WORKSHOP 1·9

Solutions

Part A

Bushfire (Forest Fire)! A bushfire is quickly burning everything in its way. Mathematicians are first on scene and estimate that the fire is spreading in the form of a rectangle. Fire crews respond and arrive after 1 hour. By this point, the mathematicians estimate that the length of the burnt rectangle is 15km and is growing at a rate of 17km/hr, and that the width is 10km and is growing at a rate of 8km/hr. How much area is burnt when they arrive? How quickly is the fire spreading at that point in time?

Solution
Let $L(t)$ and $W(t)$ be the length and width of the burnt rectangle respectively, where t is time measured in hours. Then we're given that

$$L(1) = 15 \qquad W(1) = 10$$
$$L'(1) = 17 \qquad W'(1) = 8.$$

Since the area of the burnt rectangle is just given by $A(t) = L(t)W(t)$, when the fire crews arrive, the burnt area is $A(1) = L(1)W(1) = 15 \cdot 10 = 150$ (km). The product rule tells us that

$$A'(t) = L'(t)W(t) + L(t)W'(t),$$

so, after one hour has passed, the burnt area will be growing at a rate

$$A'(1) = L'(1)W(1) + L(1)W'(1) = 17 \cdot 10 + 15 \cdot 8 = 170 + 120 = 290(\text{km/hr}).$$

Part B

Dropping like dropbears! A population of dropbears are caught in the middle of the fire are slowly moving away from the fire. However, they move too slowly and over time, the fire catches up to some of them *sad face*. The number of dropbears that are caught in the fire is dependent on the size of the fire $A(t)$. Latest mathematical reports suggests that the number of dropbears that are being roasted at time t (in hours) is given by $B(t)$, where:

$$B(t) = \frac{A(t)}{F(t)} \qquad F(t) = 5(t^2 + 1)$$

Find the number of dropbears that have met their fiery end when the fire crews arrive after 1 hour. How quickly are they dying (dropping) after at that point in time? (From the previous problem we know that $A(1) = 150$ km^2 and $A'(1) = 290$ km^2/hr

Dropbears are native to Australia. According to Wiki: *Drop bears are commonly said to be unusually large, vicious, carnivorous marsupials related to koalas (although the koala is not a bear) that inhabit treetops and attack their prey by dropping onto their heads from above.*

Solution
Recall from the last part that $A(1) = 150$ and $A'(1) = 290$. This will be handy later on. Since we are given $F(t) = 5(t2 + 1)$, we canwork out what happens at $t = 1$.

$$F(t) = 5(t2 + 1) \qquad F(1) = 10$$
$$F'(t) = 5(2t) = 10t \qquad F'(1) = 10.$$

At time t, the number of roasted dropbears is given by $B(t)$, so that means after 1 hour, we have:

$$B(t) = \frac{A(t)}{F(t)} \qquad\qquad B(1) = \frac{A(1)}{F(1)} = \frac{150}{10} = 15.2$$

We can do that since the denominator is not zero. For the derivative, we can make use of the quotient rule.

$$\begin{aligned} B'(t) &= \frac{F(t) \cdot A'(t) - F'(t) \cdot A(t)}{F(t)^2} \\ B'(1) &= \frac{F(1) \cdot A'(1) - F'(1) \cdot A(1)}{F(1)^2} \\ B'(1) &= \frac{10 \cdot 290 - 10 \cdot 150}{(10)^2} \\ B'(1) &= 14. \end{aligned}$$

That means the dropbears are getting roasted at 14 per hour.
Some things to highlight. The function $B(t) = \frac{A(t)}{F(t)} = \frac{L(t)W(t)}{F(t)}$. So what we have done here is to break $B(t)$ into three functions. To find the derivative, we applied the product rule to the numerator first and then the quotient rule to the result. What might happen if we tried to do the quotient rule first and then used the product rule? That is, writing $B(t) = L(t) \cdot \frac{W(t)}{F(t)}$?

Part C

Survival! After the fire has been put out, the fire crews find a surviving baby dropbear after confusing it with a cylinder. (It is a well known mathematical fact that animals are often confused with simple geometric objects). Latest studies showed that the height of a dropbear h(t) in centimetres after t weeks is given by:

$$h(t) = \frac{15}{2} \cdot \frac{5t + \sqrt{t} + 1}{t + 1}$$

While no such study has been done with the cross-section radius of a dropbear (because cutting them up would be unethical), empirical studies suggests that:

Time (wks)	1	2	3	4	5
Radius (cm)	8.75	10.4	11.5	12.4	13.2
Radius Growth Rate (cm/wk)	2	1.3	1	0.8	0.67

It is estimated that the baby dropbear can be released back into the wild after 4 weeks. Assuming the the bear has the density of water, how heavy will the dropbear be at that point in time? How fast is it growing at that point in time?

Solution
We have to play the same game here, except we are interested in $t = 4$, so let's figure out each component first:

$$\begin{aligned} h(t) &= \frac{15}{2} \cdot \frac{5t + \sqrt{t} + 1}{t + 1} \\ h(4) &= \frac{15}{2} \cdot \frac{5 \cdot 4 + \sqrt{4} + 1}{4 + 1} = 34 : 5cm. \end{aligned}$$

Similarly, we can find the derivative:

$$\begin{aligned} h'(t) &= \frac{(5t+\sqrt{t}+1)'(t+1) - (5t+\sqrt{t}+1)(t+1)'}{(t+1)^2} \\ &= \frac{(5+\frac{1}{2\sqrt{t}})(t+1) - (5t+\sqrt{t}+1)}{(t+1)^2} \\ h'(4) &= \frac{(5+\frac{1}{2\sqrt{4}})(4+1) - (5\cdot 4+\sqrt{4}+1)}{(4+1)^2} \\ &= 0.975\text{cm/wk}. \end{aligned}$$

Ultimately we want calculate the mass and its derivative at time $t = 4$, but since we're assuming the density of the dropbear is that of water, we just need to consider the volume of the dropbear. We can read off the table of values to get $r(4) = 12.4$ and $r'(4) = 0.8$. Then we can look at the volume of a cylinder:

$$\begin{aligned} V(t) &= \pi \cdot r(t) \cdot r(t) \cdot h(t) \\ V(4) &= \pi \cdot r(4) \cdot r(4) \cdot h(4) \\ V(4) &= \pi \cdot 12.4 \cdot 12.4 \cdot 0.975 \\ V(4) &= 5304.72\pi \approx 16665.27\text{cm}^3. \end{aligned}$$

To get the derivative, more work is required: we look at diferentiating the function:

$$\begin{aligned} V(t) &= \pi \cdot r(t) \cdot r(t) \cdot h(t) \\ V'(t) &= \pi[\cdot r(t) \cdot (r(t) \cdot h(t))]' \\ &= \pi \cdot \left[r(t) \cdot (r(t) \cdot h(t))' + r'(t) \cdot (r(t) \cdot h(t))\right]. \end{aligned}$$

We then need the product rule again

$$\begin{aligned} V'(t) &= \pi \cdot [r(t) \cdot (r'(t) \cdot h(t) + r(t) \cdot h'(t)) + r'(t) \cdot (r(t) \cdot h(t))] \\ &= \pi \cdot (r(t) \cdot r'(t) \cdot h(t) + r(t) \cdot r(t) \cdot h'(t) + r'(t) \cdot r(t) \cdot h(t)) \end{aligned}$$

Now we just substitute for $t = 4$

$$\begin{aligned} V'(4) &= \pi \cdot (r(4) \cdot r'(4) \cdot h(4) + r(4) \cdot r(4) \cdot h'(4) + r'(4) \cdot r(4) \cdot h(4)) \\ V'(4) &= \pi \cdot (12.4 \cdot 0.8 \cdot 34.5 + 12.4 \cdot 12.4 \cdot 0.975 + 0.8 \cdot 12.4 \cdot 34.5) \\ V'(4) &= \pi \cdot 834.396 \approx 2621.33\text{cm}^3\text{/wk}. \end{aligned}$$

So we can conclude that the mass of the dropbear is approximately 16.7 kg and growing at a rate of about 2.6 kg/wk.