WORKSHOP 1·7

Solution

This workshop is meant to help you review the main topics covered so far in the course, in preparation for the upcoming midterm exam.
Work out each of the following problems on the board. Make sure you discuss your solution with your TAs before you erased the board and start a new problem.

1. (a) Find the domain of

$$g(x) = \frac{x-1}{3x - \frac{9}{x}}.$$

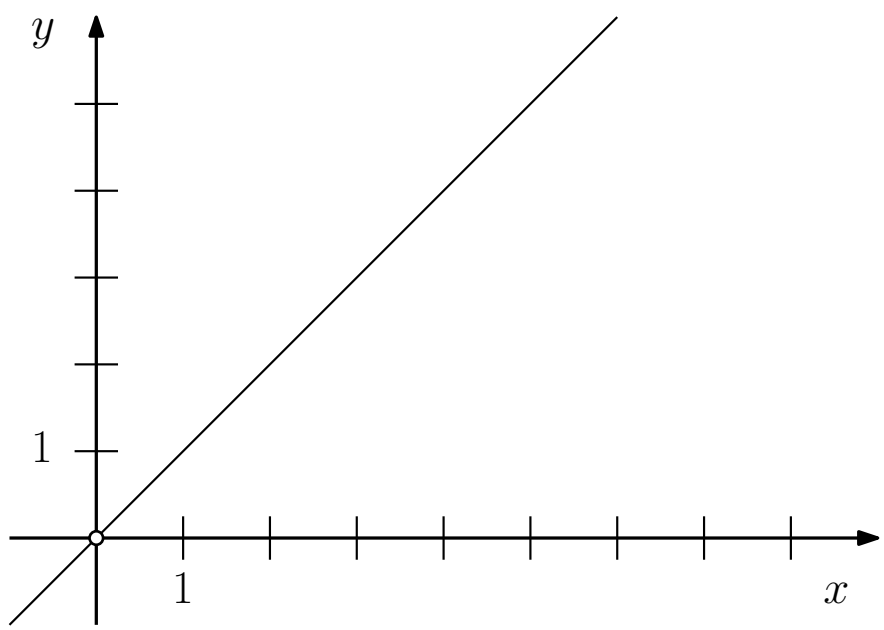
Solution
$g(x)$ is defined for all real numbers x except $x = 0$ (where $\frac{9}{x}$ is undefined) and where $3x - \frac{9}{x} = 0$. $3x - \frac{9}{x} = 0$ at $x = \pm\sqrt{3}$. So, the domain of $g(x)$ is the real numbers except for 0 and $\pm\sqrt{3}$, or $\mathbb{R} \setminus \{0, \pm\sqrt{3}\}$.

(b) Find the domain and range of $F(x) = f(f(x))$ where

$$f(x) = \frac{1}{x}.$$

Sketch the graph of $F(x)$ and determine whether F is invertible.

Solution
It is useful here to think what the composition $f(f(x))$ actually works. Generally, working from the inside out, we take the input x, then take its multiplicative inverse $\frac{1}{x}$ then take another inverse to go back to x.
The domain is the set of all possible inputs x that yield an output $f(f(x))$. All numbers except 0 have an inverse, and once you take that inverse you can always take it again to find the original input. Therefore the domain is all the real numbers except for 0, or $\mathbb{R} \setminus \{0\}$. The range is the same set, as outside of $x = 0$, $f(f(x)) = x$.
The function is thus the line $y = x$ with a hole at the origin.



To find whether F is invertible we take any element y in its image and we need to find a unique x in the domain such that $F(x) = y$. Since $F(x) = x$ in its domain, we can always use $y = x$, therefore it is invertible. We could also have used the fact that the line $y = x$ is its own inverse and $F(x)$ is the same function except that it's missing a point at the origin, but all the other points were already invertible, so $F(x)$ is invertible. A third way to show this is that doing a reflexion of the previous graph on the $y = x$ line also yields a function (the shape of the graph does not change this time) therefore $F(x)$ is invertible.

2. An outdoor hot tub holds $4000L$ of water. If a small valve at the bottom of the tub is opened, then the volume of water in the tub is modelled by the function

$$V(t) = 4000(1 - t)^2,$$

where V is the volume of water in the hot tub, in *litres*, and t is the time, in *hour*, since the valve is open.

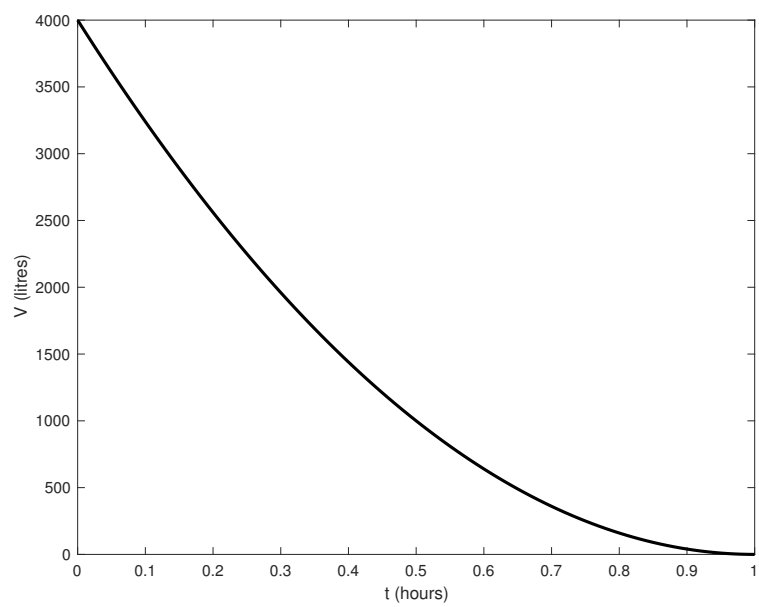
(a) How long does it take for the water to completely drain?

Solution

The hot tub is empty when $V(t) = 0$, which occurs at $t = 1$. The hot tub starts draining at $t = 0$. So, it takes 1 *hour* for the hot tub to drain.

(b) Sketch the graph of the volume function. (Note that by part (a) there would be no water in the tub after some point in time.) Make sure you indicate the value of any intercept on your sketch. Your diagram does not need to be in scale.

Solution



(c) Find the average rate of change in the volume of the water during the first half hour. Include units in your answer.

Solution

The average rate of change in the volume of the water is the change in the volume of the water per the change in time. Over the time from $t = 0$ to $t = \frac{1}{2}$, the average rate of change in the volume of the water is

$$\frac{4000(1-\frac{1}{2})^2 - 4000(1-0)^2}{\frac{1}{2}-0}\frac{litres}{hour} = -6000\frac{litres}{hour}.$$

(d) Using the definition of instantaneous rate of change as a limit, compute the instantaneous rate of change of V at $t = 1/2$ hour.

Solution
At $t = \frac{1}{2}$ hour,

$$\frac{dV}{dt} = \lim_{h\to 0}\frac{V(\frac{1}{2}+h) - V(\frac{1}{2})}{h} = \lim_{h\to 0}\frac{4000(1-(\frac{1}{2}+h))^2 - 4000(1-\frac{1}{2})^2}{h} =$$
$$\lim_{h\to 0}\frac{4000(-h+h^2)}{h} = -4000\frac{litres}{hour}$$

3. (i) Evaluate the limit, $\lim_{x\to 1}\frac{3x+1}{2x-1}$, if it exist.

Solution
We are taking the limit of a function here and at $x = 1$ the denominator is not zero, so we may use direct substitution here:

$$\lim_{x\to 1}\frac{3x+1}{2x-1} = \frac{3\cdot 1+1}{2\cdot 1-1} = 4$$

(ii) The function $f(x)$ is defined as

$$f(x) = \begin{cases} x^2 - 3x + 4, & 0 \le x < 2 \\ g(x), & 2 \le x \le 5 \end{cases}$$

where g is an unknown function. Find an appropriate function g such that $\lim_{x\to 2} f(x) = 2$.

Solution
$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-} x^2 - 3x + 4 = 2$. For $\lim_{x\to 2} f(x)$ to exist, $\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$, which occurs if $\lim_{x\to 2^+} g(x) = 2$, as $\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} g(x)$. Any function g such that $\lim_{x\to 2^+} g(x) = 2$ will suffice. A couple of the possible functions for g are $g(x) = 2$ and $g(x) = x$.

(iii) Evaluate the limit, $\lim_{x\to 1^+}\frac{x^2-2}{x^2-4x+3}$, if it exist.

Solution
This time we are taking the limit of a fraction at a point where the denominator would be zero, but not the numerator. We may factorize the denominator as $(x-1)(x-3)$, so to the right of $x = 1$, the denominator will be a very small, but negative number. we may then evaluate the limit

$$\lim_{x\to 1^+}\frac{x^2-2}{x^2-4x+3} = \lim_{x\to 1^+}\frac{x^2-2}{(x-1)(x-3)}$$
$$= \frac{(-)}{(+,\ \text{close to } 0)(-)} = +\infty$$

4. Find a value of the constant a such that $\lim_{x \to a} f(x)$ exists if

$$f(x) = \begin{cases} x^2 - 3x - 18, & x < a \\ 0, & x = a \\ -2 - 3x, & x > a \end{cases}$$

Is f continuous at $x = a$ (for a equal to the value you found above)?

Solution

Taking the one sided limits at $x = a$ we obtain:

$$\begin{aligned} \lim_{x \to a^-} f(x) &= \lim_{x \to a^-} x^2 - 3x - 18 \\ &= a^2 - 3a - 18 \\ \lim_{x \to a^+} f(x) &= \lim_{x \to a^+} -2 - 3x \\ &= -2 - 3a. \end{aligned}$$

For the limit to exist, we need the two limits to be equal, therefore we may solve for a:

$$\begin{aligned} a^2 - 3a - 18 &= -2 - 3a \\ a^2 - 16 &= 0 \\ a &= \pm 4. \end{aligned}$$

Thus either $a = -4$ or $a = 4$ works but in both cases f will not be continuous because the limit we find will be either -8 or 10, which doesn't match the actual value of the function, which is 0.

5. Carefully prove there is at least one solution to the equation $x^4 - x^3 + 2x^2 - 1 = 0$ between $x = -1$ and $x = 1$. Is there more than one solution?

Solution

Note that the question does not explicitly asks to find the value of the solution. This is an indication that the Intermediate Value Theorem is key to the proof.

We already know that $f(x) = x^4 - x^3 + 2x^2 - 1$ is a continuous function everywhere because it's a polynomial. Let's try evaluating the function at some easy points:

$$\begin{aligned} f(-1) &= (-1)^4 - (-1)^3 + 2(-1)^2 - 1 = 3 > 0 \\ f(0) &= (0)^4 - (0)^3 + 2(0)^2 - 1 = -1 < 0 \\ f(1) &= (1)^4 - (1)^3 + 2(1)^2 - 1 = 1 > 0. \end{aligned}$$

We then know that a continuous function goes from the value of 3 at $x = -1$ to -1 at $x = 0$, then the intermediate value theorem tells us that there must be a root c, that is $f(c) = 0$ between -1 and 0.

We may repeat this process in the interval from $x = 0$ to $x = 1$. We have a continuous function going from -1 to 1 over the interval, so it must reach the intermediate value of 0 at some point d between 0 and 1.

6. Find the equation of the tangent line to the curve $y = x^2 - 2x + 2$ at the point with x-coordinate $x = 3$.

Solution

To find the equation of the tangent line we will use the point-slope formula. The point is given by the value of the function at $x = 3$, since the tangent line must touch the curve of the graph exactly there:

$$y = 3^2 - 2 \cdot 3 + 2 = 5.$$

Therefore the line must cross the point $(3, 5)$.

To find the slope of the tangent line we need to find the derivative of $g(x) = x^2 - 2x + 2$ at $x = 2$. There are multiple ways to do this. We may use the limit definition:

$$\begin{aligned}
g'(3) &= \lim_{h \to 0} \frac{g(3+h) - g(3)}{h} \\
&= \lim_{h \to 0} \frac{(3+h)^2 - 2(3+h) + 2 - (3^2 - 2 \cdot 3 + 2)}{h} \\
&= \lim_{h \to 0} \frac{(3^2 + 6h + h^2) - (6 + 2h) - (9 - 6)}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 4h}{h} \\
&= \lim_{h \to 0} h + 4 = 4.
\end{aligned}$$

We then need the equation of the line of slope 4 passing through the point $(3, 5)$. Using the point-slope formula we obtain the equation of the tangent line:

$$y = 4(x - 3) + 5.$$

The graph below shows the graph of the function and its tangent line, generated by computer.