WORKSHOP 1·7

Solution

This workshop is meant to help you review the main topics covered so far in the course, in preparation for the upcoming midterm exam.
Work out each of the following problems on the board. Make sure you discuss your solution with your TAs before you erased the board and start a new problem.

1. Find the domain and range of $F(x) = f(f(x))$ where

$$f(x) = \frac{1}{x}.$$

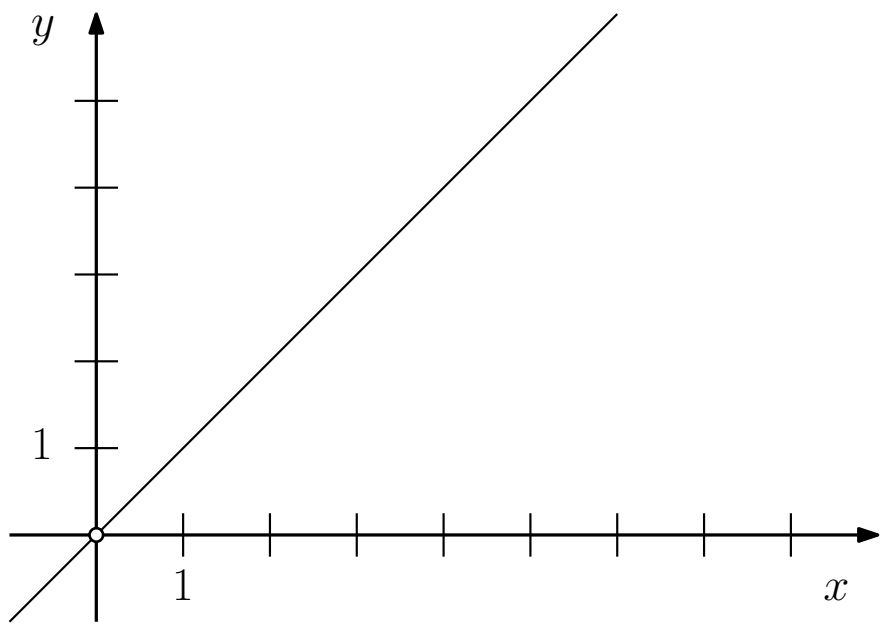
Sketch the graph of $F(x)$ and determine whether F is invertible.

Solution

It is useful here to think what the composition $f(f(x))$ actually works. Generally, working from the inside out, we take the input x, then take its multiplicative inverse $\frac{1}{x}$ then take another inverse to go back to x.

The domain is the set of all possible inputs x that yield an output $f(f(x))$. All numbers except 0 have an inverse, and once you take that inverse you can always take it again to find the original input. Therefore the domain is all the real numbers except for 0, or $\mathbb{R} \setminus \{0\}$. The range is the same set, as outside of $x = 0$, $f(f(x)) = x$.

The function is thus the line $y = x$ with a hole at the origin.

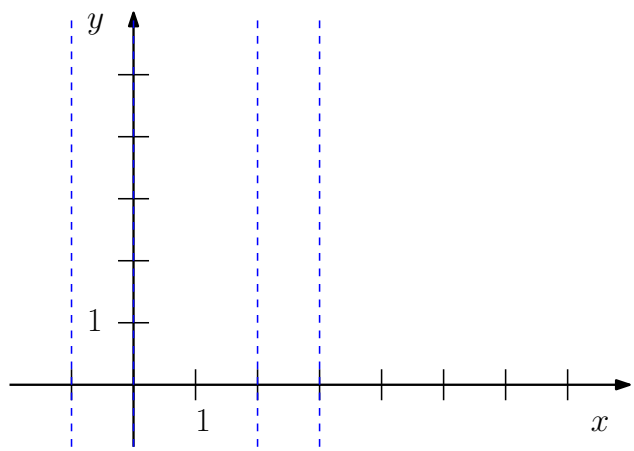


To find whether F is invertible we take any element y in its image and we need to find a unique x in the domain such that $F(x) = y$. Since $F(x) = x$ in its domain, we can always use $y = x$, therefore it is invertible. We could also have used the fact that the line $y = x$ is its own inverse and $F(x)$ is the same function except that it's missing a point at the origin, but all the other points were already invertible, so $F(x)$ is invertible. A third way to show this is that doing a reflexion of the previous graph on the $y = x$ line also yields a function (the shape of the graph does not change this time) therefore $F(x)$ is invertible.

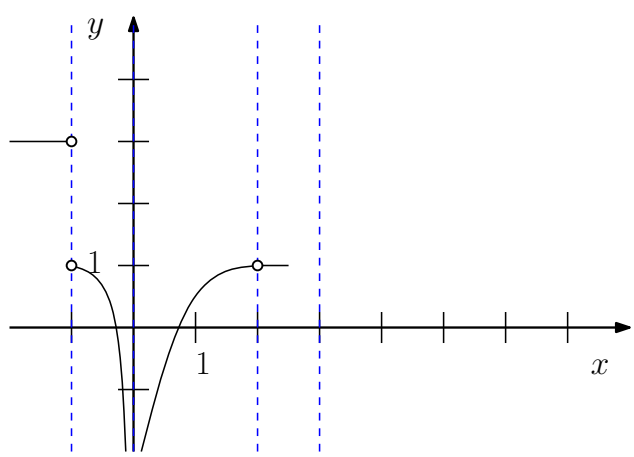
2. Sketch the graph of a function $f(x)$ whose domain is all real numbers except 0, and such that $\lim_{x\to -1^+} f(x) = 1$, $\lim_{x\to -1^-} f(x) = 3$, $\lim_{x\to 0} f(x) = -\infty$, $\lim_{x\to 2} f(x) = 1$, and the slope of the tangent line to the curve at (3,2) is -1.

Solution

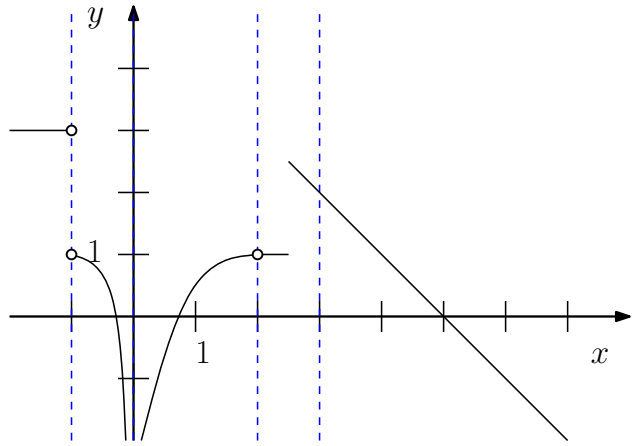
It is always good practice to draw these functions step by step. We first begin with only the axes and the x coordinates specified in the problem statement, that is $x = -1, 0, 2,$ and 3.



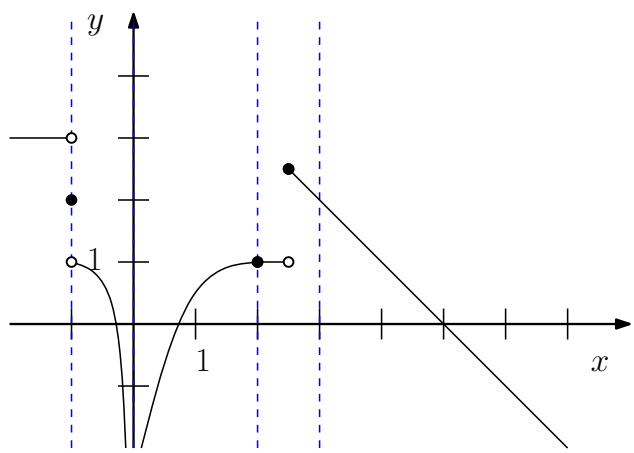
We then mark on the graph the coordinates of each limits and trace the function in the appropriate direction. At $x = -1$, the function should approach the value of $y = 3$ from the left, so the constant function $y = 3$ works left of $x = -1$. To the right it must approach $y = 1$ and then go down to $-\infty$ when getting close to $x = 0$ so we trace the function starting at the point $(-1, 1)$ and then goes down to negative infinity, like the function $\frac{1}{x}$ to the left of $x = 0$. Since the limit at zero is a full limit, it needs to be so from both sides, so the same $\frac{1}{x}$ shape is also present to the right of $x = 0$ and then reaches the point $(1, 2)$ to satisfy the limit at $x = 2$. That limit is also a full limit so we add a small segment through the last point to show the full limit is satisfied.



We then need to make sure that the slope of the tangent line at $(3, 2)$ is -1, so we add that point on the graph and a segment of a line of slope -1 through it.



We then have a function which satisfies all of the limits and tangent line slope requirements. The last thing we need is to make sure that the domain is all nonzero real numbers. If we extend the function to the left of $x = -1$ and to the right of $x = 3$ all those x values are accounted for. Starting from the left, the first point that is not yet defined is at $x = -1$. The problem statement doesn't specify anything about the function exactly at $x = -1$, so we may add a point anywhere on the $x = -1$ line. The next missing point is at $x = 0$, but it's not in the domain, so we leave it empty. Then comes $x = 2$ and once again we may fill anywhere on the vertical line and finally we need to settle what happens between $x = 2$ and $x = 3$. We can just add another jump and make sure that each x value has a corresponding y value, which leads to a common, disk-circle jump.



We finally have a function that satisfies all of the conditions stated in the problem. It can be useful to verify each condition after the final draft.

3. Evaluate the following limits, if they exist.

(i) $\lim_{x \to 1} \frac{3x+1}{2x-1}$

Solution

We are taking the limit of a function here and at $x = 1$ the denominator is not zero, so we may use direct substitution here:

$$\lim_{x \to 1} \frac{3x+1}{2x-1} = \frac{3 \cdot 1 + 1}{2 \cdot 1 - 1} = 4$$

(ii) $\lim_{x \to 2} f(x)$ where

$$f(x) = \begin{cases} x^2 - 3x + 4, & 0 \leq x \leq 2 \\ 8 - 3x, & 2 < x \leq 5 \end{cases}$$

Solution

We are taking a limit at the breaking point of a piecewise function, so we have to take one-sided limits:

$$\begin{aligned} \lim_{x \to 2^-} f(x) &= \lim_{x \to 2^-} x^2 - 3x + 4 \\ &= 2^2 - 3 \cdot 2 + 4 = 2 \\ \lim_{x \to 2^+} f(x) &= \lim_{x \to 2^+} 8 - 3x \\ &= 8 - 3 \cdot 2 = 2 \end{aligned}$$

The two limits are equal, therefore the full limit exists and is equal to 2.

(iii) $\lim_{x \to 1^+} \frac{x^2 - 2}{x^2 - 4x + 3}$

Solution

This time we are taking the limit of a fraction at a point where the denominator would be zero, but not the numerator. We may factorize the denominator as $(x-1)(x-3)$, so to the right of $x = 1$, the denominator will be a very small, but negative number. we may then evaluate the limit

$$\begin{aligned} \lim_{x \to 1^+} \frac{x^2 - 2}{x^2 - 4x + 3} &= \lim_{x \to 1^+} \frac{x^2 - 2}{(x-1)(x-3)} \\ &= \frac{(-)}{(+, \text{ close to } 0)(-)} = +\infty \end{aligned}$$

4. Find a value of the constant a such that $\lim_{x \to a} f(x)$ exists if

$$f(x) = \begin{cases} x^2 - 3x - 18, & x < a \\ 0, & x = a \\ -2 - 3x, & x > a \end{cases}$$

Is f continuous at $x = a$ (for a equal to the value you found above)?

Solution

Taking the one sided limits at $x = a$ we obtain:

$$\begin{aligned} \lim_{x \to a^-} f(x) &= \lim_{x \to a^-} x^2 - 3x - 18 \\ &= a^2 - 3a - 18 \\ \lim_{x \to a^+} f(x) &= \lim_{x \to a^+} -2 - 3x \\ &= -2 - 3a. \end{aligned}$$

For the limit to exist, we need the two limits to be equal, therefore we may solve for a:

$$\begin{aligned} a^2 - 3a - 18 &= -2 - 3a \\ a^2 - 16 &= 0 \\ a &= \pm 4. \end{aligned}$$

Thus either $a = -4$ or $a = 4$ works but in both cases f will not be continuous because the limit we find will be either -8 or 10, which doesn't match the actual value of the function, which is 0.

5. Carefully prove there is at least one solution to the equation $x^4 - x^3 + 2x^2 - 1 = 0$ between $x = -1$ and $x = 1$. Is there more than one solution?

Solution

Note that the question does not explicitly asks to find the value of the solution. This is an indication that the Intermediate Value Theorem is key to the proof.

We already know that $f(x) = x^4 - x^3 + 2x^2 - 1$ is a continuous function everywhere because it's a polynomial. Let's try evaluating the function at some easy points:

$$\begin{aligned} f(-1) &= (-1)^4 - (-1)^3 + 2(-1)^2 - 1 = 3 > 0 \\ f(0) &= (0)^4 - (0)^3 + 2(0)^2 - 1 = -1 < 0 \\ f(1) &= (1)^4 - (1)^3 + 2(1)^2 - 1 = 1 > 0. \end{aligned}$$

We then know that a continuous function goes from the value of 3 at $x = -1$ to -1 at $x = 0$, then the intermediate value theorem tells us that there must be a root c, that is $f(c) = 0$ between -1 and 0.

We may repeat this process in the interval from $x = 0$ to $x = 1$. We have a continuous function going from -1 to 1 over the interval, so it must reach the intermediate value of 0 at some point d between 0 and 1.

6. Find the equation of the tangent line to the curve $y = x^2 - 2x + 2$ at the point with x-coordinate $x = 3$.

Solution

To find the equation of the tangent line we will use the point-slope formula. The point is given by the value of the function at $x = 3$, since the tangent line must touch the curve of the graph exactly there:

$$y = 3^2 - 2 \cdot 3 + 2 = 5.$$

Therefore the line must cross the point $(3, 5)$.

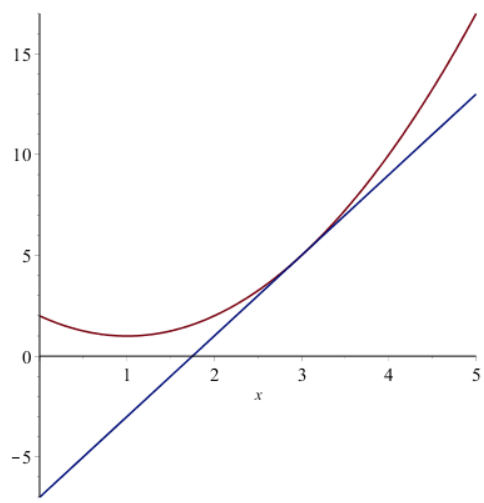
To find the slope of the tangent line we need to find the derivative of $g(x) = x^2 - 2x + 2$ at $x = 2$. There are multiple ways to do this. We may use the limit definition:

$$\begin{aligned}
g'(3) &= \lim_{h \to 0} \frac{g(3+h) - g(3)}{h} \\
&= \lim_{h \to 0} \frac{(3+h)^2 - 2(3+h) + 2 - (3^2 - 2 \cdot 3 + 2)}{h} \\
&= \lim_{h \to 0} \frac{(3^2 + 6h + h^2) - (6 + 2h) - (9 - 6)}{h} \\
&= \lim_{h \to 0} \frac{h^2 + 4h}{h} \\
&= \lim_{h \to 0} h + 4 = 4.
\end{aligned}$$

We then need the equation of the line of slope 4 passing through the point $(3, 5)$. Using the point-slope formula we obtain the equation of the tangent line:

$$y = 4(x - 3) + 5.$$

The graph below shows the graph of the function and its tangent line, generated by computer.



7. Sketch the parabolas $y = x^2$ and $y = x^2 - 2x + 2$. Is there a line that is tangent to both curves? If so, find its equation. If not, why not?

Solution

First of all the graph or the two functions look as in figure 1 below. We clearly see that there must be a common tangent line to the two functions and is pictured in brown.

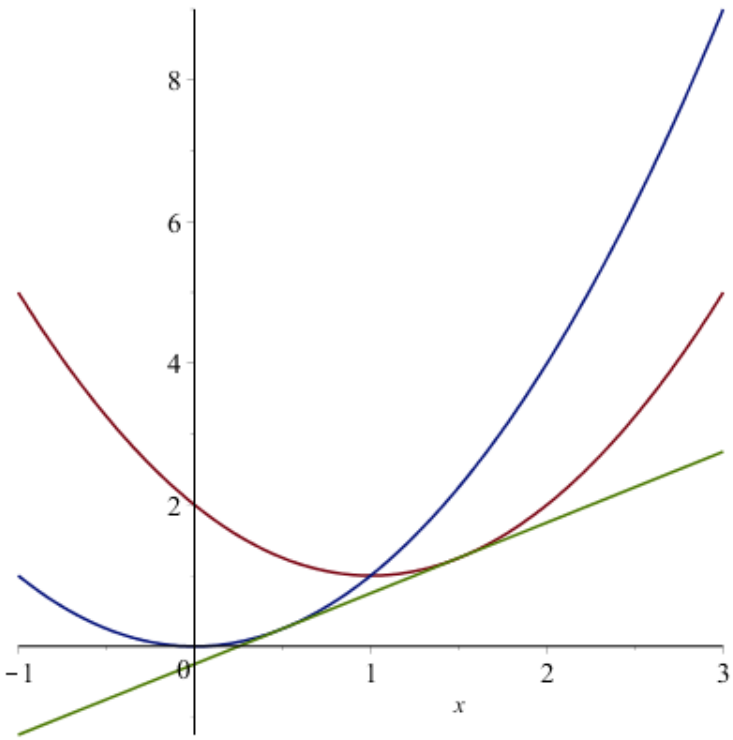


Figure 1: Graph of the two parabolas and their common tangent line.

Then we recall the formula of the tangent line to a function $f(x)$ at a point a:

$$L(x) = f'(a)(x - a) + f(a) = f'(a)x + [f(a) - af'(a)].$$

Define the two functions f and g as $f(x) = x^2$ and $g(x) = x^2 - 2x + 2$. We will take the tangent line at a for f and at b for g.

For two lines to be equal they need to have the same slope and intersect:

$$f'(a) = g'(b), \quad f(a) - af'(a) = g(b) - bg'(b).$$

We thus have two equations and two unknowns and can solve for a and b. We first make the two slopes equal:

$$\begin{aligned} f'(a) &= g'(b) \\ 2a &= 2b - 2 \\ a &= b - 1. \end{aligned}$$

We then use this result in the second equation:

$$\begin{aligned}
f(a) - af'(a) &= g(b) - bg'(b) \\
a^2 - 2a^2 &= (b^2 - 2b + 2) - (2b^2 - 2b) \\
-(b-1)^2 &= -b^2 + 2 \\
-b^2 + 2b - 1 &= -b^2 + 2 \\
2b &= 3 \\
b &= \frac{3}{2} \\
a &= \frac{1}{2}.
\end{aligned}$$

Using this in the tangent line formula we find that the line

$$L(x) = f'(a)(x-a) + f(a) = \left(x - \frac{1}{2}\right) + \frac{1}{4}$$

is tangent to both functions.